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ALMOST FLAT MANIFOLDS

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1. Introduction

1.1. We denote by V a connected n-dimensional complete Riemannian manifold, by $d = d(V)$ the diameter of V, and by $c^+ = c^+(V)$ and $c^- = c^-(V)$, respectively, the upper and lower bounds of the sectional curvature of V. We set $c = c(V) = \max(|c^+|, |c^-|)$.

We say that V is ε-flat, $\varepsilon \geq 0$, if $cd^2 \leq \varepsilon$.

1.2. Examples.

a. Every compact flat manifold is ε-flat for any $\varepsilon \geq 0$.

b. Every compact nil-manifold possesses an ε-flat metric for any $\varepsilon \geq 0$. (*A* manifold is called a nil-manifold if it admits a transitive action of a nilpotent Lie group; see 4.5.)

The second example shows that for $n \geq 3$, $\varepsilon > 0$ there are infinitely many ε-flat n-dimensional manifolds with different fundamental groups.

1.3. Define inductively $ex_i(x) = \exp(ex_{i-1}(x))$, $ex_0(x) = x$, and set $\hat{\varepsilon}(n) = \exp(-ex_j(n))$, where $j = 200$. (We are generous everywhere in this paper because the true value of the constants is unknown.)

1.4. Main Theorem. *Let V be a compact $\hat{\varepsilon}(n)$-flat manifold, and π its fundamental group. Then*:

(a) *There exists a maximal nilpotent normal divisor $N \subset \pi$*;

(b) $\operatorname{ord}(\pi/N) \leq ex_3(n)$;

(c) *the finite covering of V corresponding to N is diffeomorphic to a nil-manifold.*

Corollary. *If V is $\hat{\varepsilon}(n)$-flat, then its universal covering is diffeomorphic to R^n. If V is $\hat{\varepsilon}(n)$-flat and π is commutative, then V is diffeomorphic to a torus.*

1.5. Manifolds of positive and almost positive curvature. For such manifolds one expects the properties (a) and (b) from Main theorem 1.4, but we are able to prove only the following:

(i) If V is a manifold of nonnegative sectional curvature ($c^- \geq 0$), then its fundamental group π and every subgroup of π can be generated by 3^n elements.

(ii) If $d(V) \leq \mathscr{D}$, $c^-(V) \geq -K$, $K \geq 0$, then π can be generated by $N \leq 3^n\, ex_2(nK\mathscr{D}^2)$ elements; if π is a free group and $K\mathscr{D}^2 \leq \hat{\varepsilon}(n)$, then π is generated by one element.

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