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MANIFOLDS OF NEGATIVE CURVATURE

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1. Statement of results

1.1. For a Riemannian manifold V we denote by $c^+(V)$ and $c^-(V)$ respectively the upper and the lower bounds of the sectional curvature, by vol (V) the volume, and by $d(V)$ the diameter.

1.2. Let V be an n-dimensional closed Riemannian manifold of negative curvature and $c^-(V) \geq -1$. If $n \geq 8$, then vol $(V) \geq C(1 + d(V))$, where the constant $C > 0$ depends only on n.

Remark. This inequality is exact: For each n there exists an infinite sequence V_i with $d(V_i) \to \infty$, $i \to \infty$, and with uniformly bounded ratio vol $(V_i)/d(V_i)$.

Proof. Take a manifold V of constant negative curvature with infinite group $H_1(V)$ (see [8]) and a sequence of its finite cyclic coverings.

For $n = 4, 5, 6, 7$ we shall prove here the following weaker result: vol $(V) \geq C(1 + d^{1/3}(V))$. Notice that arguments from § 4 show that for $n \geq 4$ an n-dimensional manifold V with $-\varepsilon \geq c^+(V) \geq c^-(V) \geq -1$, $\varepsilon > 0$, satisfies: vol $(V) \geq C(1 + d(V))$ where C depends on n and ε.

1.3. Theorem 1.2 sharpens the Margulis-Heintze theorem (see [6], [4]) stating the inequality vol $(V) \geq C = C_n$. In this paper we prove the following generalization.

1.3A. Let X be a complete simply connected manifold of negative curvature with $c^-(X) > -1$. Let Γ be a discrete group (possibly with torsion) of isometries of V. Then vol $(X/\Gamma) \geq C$, where $C > 0$ depends only on dim (X).

This fact is still true for manifolds of nonpositive curvature with $c^-(X) \geq -1$ and negative Ricci curvature (see [5]). In the homogeneous case this is the Kazhdan-Margulis theorem (see [9]).

The finiteness theorems

1.4. Combining § 1.2 with Cheeger's results (see [1], [4]) we immediately conclude:

For given $n \neq 3$ and $C > 0$ there exist only finitely many pairwise non-diffeomorphic closed n-dimensional manifolds V with $0 > c^+(V) \geq c^-(V) > -1$ and vol $(V) \leq C$.

1.5. Counter-example for $n = 3$. There exists an infinite sequence of 3-di-

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