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INTEGRAL INVARIANTS OF CONVEX CONES

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Introduction

Let E be an $(n + 1)$-dimensional real vector space, E^* its dual space,' K a convex nondegenerate pointed cone in E, and K^* the dual cone in E^*. It is our main purpose to study geometric objects in K (and K^*) from the viewpoint of invariance under transformations of the general linear group $GL(n + 1, R)$, and of the unimodular group $SL(n + 1, R)$. So, this matter will occupy most of the present work. However, since there is a natural correspondence between flat bounded cross-sections of $(n + 1)$-dimensional convex cones and n-dimensional convex bodies, our first chapter will be somehow diverse from that main object.

More precisely, if B is a convex body with nonempty interior, relative to an n-dimensional affine space F, we can imbed F in E as a hyperplane not passing through the origin and define a convex nondegenerate pointed cone in E by

$$K(B) = \{\lambda X : X \in B, \lambda > 0\} .$$

Conversely, given E, E^*, K and K^* as above, for each nonzero $\mathscr{X} \in E^*$ we can define a hyperplane $P_{\mathscr{X}} \subset E$ by

$$P_{\mathscr{X}} = \{X : X \in E, \mathscr{X} \cdot X = 1\} .$$

$P_{\mathscr{X}} \cap K$ is a convex body with nonempty interior, relative to the n-dimensional affine space $P_{\mathscr{Y}}$ if and only if $\mathscr{X} \in \mathrm{Int}\,(K^*)$. This correspondence suggests a connection between geometrical properties of $(n + 1)$-dimensional convex cones and those of n-dimensional convex bodies.

In § 1 we study a class of real valued functionals on the set of convex bodies with nonempty interior, relative to an n-dimensional affine space F; these functionals will be invariant under the action of the group $AGL(n, R)$ of all affine transformations as acting on F, and one of them, which we shall call the mean square fractional volume, will play a fundamental role in the sections which follow.

The volume of truncated cones in K can be expressed in a natural way by

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