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LINEARLY INDUCED VECTOR FIELDS AND R^2-ACTIONS ON SPHERES

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1. Introduction

We prove here results on the generic and structurally stable properties of linearly induced vector fields and R^2-actions on spheres. These actions are obtained from linear actions on R^n which are naturally extended to the standard sphere S^n via central projection. Similarly, one can use radial projection to get quite a large number of vector fields and R^2-actions on spheres which are structurally stable or at least Ω-stable.

In 1881 Poincaré [12] began the qualitative theory of polynomial vector fields on the plane R^2 looking at the central projection of their trajectories on the sphere S^2. This work appears in other texts [3], [6], [11], [13] always in a form similar to the original one. More recently Gonzalez [5] characterized the polynomial vector fields on R^2 which are structurally stable in a neighborhood of infinity. He also began the study of linearly induced vector fields on S^3.

In § 2 we consider linearly induced vector fields on the sphere S^n. Let $X(x) = Ax$ be a linear vector field on R^n. The central projection is the map which associates to each point $x = (x_1, \cdots, x_n)$ of R^n two points in S^n, $f(x) = (x_1, \cdots, x_n, 1)/\Delta x$ and $f_1(x) = -(x_1, \cdots, x_n, 1)/\Delta x$ where $\Delta x = (1 + x_1^2 + \cdots + x_n^2)^{1/2}$. The linearly induced vector fields $Df(X)$ and $Df_1(X)$ extend naturally to the whole S^n, and one gets a vector field called the Poincaré vector field $\pi(X)$. Let $\pi_\infty(X)$ be its restriction to the equator S^{n-1} which is an invariant set. The radial projection $\tau\colon R^n - 0 \to S^{n-1}$, $\tau(x) = x/|x|$, also induces a vector field $D\tau(X)$ on the sphere S^{n-1}.

Theorem 1. *Let $\pi(X)$, $X(x) = Ax$, be a Poincaré vector field on S^n. Then $\pi(X)$ is a Morse-Smale vector field if and only if the eigenvalues of A have distinct (except for pairs of conjugate complex eigenvalues) nonzero real parts.*

Let $\pi(\mathscr{X})$ be the set of Poincaré vector fields on S^n with the C^r-topology, $r \geq 1$, and $\Sigma \subset \pi(\mathscr{X})$ the subset of structurally stable ones. In Theorem 2 we prove that the Morse-Smale Poincaré vector fields on S^n form an open and dense set in $\pi(\mathscr{X})$ which coincides with Σ.

Similar results hold for linearly induced vector fields by radial projection, as shown in Theorems 3 and 4.

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