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ON THE THEORY OF NORMAL VARIATIONS

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1. Introduction

Let M^n be an n-dimensional submanifold of a Riemannian manifold M^m. An infinitesimal deformation of M^n in M^m along a normal vector field ξ is called a normal variation. In this paper we shall study some fundamental properties of nomal variations.

In § 3 we shall prove that the submanifold M^n is totally geodesic (respectively, totally umbilical or minimal) if and only if every normal variation of M^n is isometric (respectively, conformal or volume-preserving). In § 4 we shall prove that the normal variation given by ξ is affine if and only if the second fundamental tensor with respect to ξ is parallel. In § 5 we shall show that the normal variation given by ξ carries a totally geodesic (respectively, totally umbilical or minimal) submanifold into a totally geodesic (respectively, totally umbilical or minimal) submanifold when and only when ξ satisfies certain second order differential equations. In the last section, we shall study H-variations and H-stable submanifolds, and obtain a characterization of H-stable submanifolds with some applications; for example, we prove that an H-stable submanifold of a positively curved manifold has parallel mean curvature vector if and only if the submanifold is minimal.

2. Preliminaries, [1]

Let M^m be an m-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$, and denote by g_{ji}, Γ^h_{ji}, ∇_j, $K_{kji}{}^h$, K_{ji} and K the metric tensor, the Christoffel symbols formed with g_{ji}, the operator of covariant differentiation with respect to Γ^h_{ji}, the curvature tensor, the Ricci tensor and the scalar curvature of M^m respectively, where and in the sequel, the indices $h, i, j, k, \cdots$ run over the range $\{\bar{1}, \bar{2}, \cdots, \bar{m}\}$.

Let M^n be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{V; y^a\}$, and denote by g_{cb}, Γ^a_{cb}, ∇_c, $K_{dcb}{}^a$, K_{cb} and K' the corresponding quantities of M^n, where and in the sequel the indices $a, b, c, d, \cdots$ run over the range $\{1, 2, \cdots, n\}$.

Suppose that M^n is isometrically immersed in M^m by the immersion $i: M^n \longrightarrow M^m$, and identify $i(M^n)$ with M^n. Represent the immersion by

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