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A NOTE ON A THEOREM OF NIRENBERG

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The abstract forms of the nonlinear Cauchy-Kowalewski theorem are investigated in [1] and [2] in a little different formulations. We note here that the Nirenberg's formulation and proof in [1] can be simplified to give an improved abstract nonlinear Cauchy-Kowalewski theorem in a scale of Banach spaces, which contains both theorems in [1] and [2]. The proof follows that of Nirenberg exactly except one point.

Definition. Let $S = \{B_\rho\}_{\rho>0}$ be a scale of Banach spaces, and let all B_ρ for $\rho > 0$ be linear subspaces of B_0. It is assumed that $B_\rho \subset B_{\rho'}$, $\|\cdot\|_{\rho'} \leq \|\cdot\|_\rho$ for any $\rho' \leq \rho$, where $\|\cdot\|_\rho$ denotes the norm in B_ρ.

Consider in S the initial value problem of the form

$$\frac{du}{dt} = F(u(t), t) , \qquad |t| < \delta , \tag{1}$$

$$u(0) = 0 . \tag{2}$$

Assume the following conditions on F:

(i) For some numbers $R > 0$, $\eta > 0$, $\rho_0 > 0$ and every pair of numbers ρ, ρ' such that $0 \leq \rho' < \rho < \rho_0$, $(u, t) \to F(u, t)$ is a continuous mapping of

$$\{u \in B_\rho ; \|u\|_\rho < R\} \times \{t ; |t| < \eta\} \text{ into } B_{\rho'} . \tag{3}$$

(ii) For any $\rho' < \rho < \rho_0$ and all $u, v \in B_\rho$ with $\|u\|_\rho < R$, $\|v\|_\rho < R$, and for any t, $|t| < \eta$, F satisfies

$$\|F(u, t) - F(v, t)\|_{\rho'} \leq C \|u - v\|_\rho / (\rho - \rho') , \tag{4}$$

where C is a constant independent of t, u, v, ρ or ρ'.

(iii) $F(0, t)$ is a continuous function of t, $|t| < \eta$ with values in B_ρ for every $\rho < \rho_0$ and satisfies, with a fixed constant K,

$$\|F(0, t)\|_\rho \leq K / (\rho_0 - \rho) , \qquad 0 \leq \rho < \rho_0 . \tag{5}$$

Theorem. *Under the preceding hypotheses there is a positive constant a such that there exists a unique function $u(t)$ which, for every positive $\rho < \rho_0$ and $|t| < a(\rho_0 - \rho)$, is a continuously differentiable function of t with values in B_ρ, $\|u(t)\|_\rho < R$, and satisfies* (1), (2).

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