GEOMETRY OF HOROSPHERES

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1. Introduction

Let $M$ be a Hadamard manifold, i.e., a connected, simply connected, complete riemannian manifold of nonpositive curvature. To be more precise, assume that the sectional curvature $K$ of $M$ satisfies $-b^2 \leq K \leq -a^2$, where $0 < a < \infty$ and $0 < b < \infty$. If $p \in M$ and $z$ is a point at infinity (cf. Eberlein-O'Neill [4], which we give as a general reference for Hadamard manifolds), there exists a horosphere through $p$ with center $z$. This is defined as follows: Denote the geodesic ray from $p$ to $z$ by $\gamma$, and consider the geodesic spheres through $p$ with center $\gamma(t)$, $t > 0$. As $t$ goes to infinity, these spheres converge to the horosphere. More precisely, the horospheres are the level surfaces of the Busemann function $F = \lim F_t$, where $F_t$ is defined by $F_t(p) = d(p, \gamma(t)) - t$. In the flat case ($a = b = 0$), horospheres are just affine hyperplanes, and in the case of constant negative curvature, using the Poincaré model we see that horospheres are euclidean spheres internally tangent to the boundary sphere, minus the point of tangency. The main purpose of this paper is to show that, to a certain extent, the geometry of horospheres in $M$ may be compared with that in the spaces of constant curvature $-a^2$ and $-b^2$, respectively. We give two examples:

1. (Theorem 4.6). If $\mathcal{H}$ is a horosphere and $h$ denotes the distance in $\mathcal{H}$ with respect to the induced metric, then for all $p, q \in \mathcal{H}$
   \[
   \frac{2}{a} \sinh \frac{a}{2} d(p, q) \leq h(p, q) \leq \frac{2}{b} \sinh \frac{b}{2} d(p, q),
   \]
   where $d$ is the distance function of $M$.

2. (Theorem 4.9). If $\gamma$ is a geodesic tangent to a horosphere $\mathcal{H}$, and if $p, q$ are the projections of $\gamma(\pm \infty)$ onto $\mathcal{H}$, then
   \[
   \frac{2}{b} \leq h(p, q) \leq \frac{2}{a}.
   \]

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