J. DIFFERENTIAL GEOMETRY
12 (1977) 481-491

GEOMETRY OF HOROSPHERES

ERNST HEINTZE & HANS-CHRISTOPH IM HOF

1. Introduction

Let M be a Hadamard manifold, i.e., a connected, simply connected, complete riemannian manifold of nonpositive curvature. To be more precise, assume that the sectional curvature K of M satisfies $-b^2 \leq K \leq -a^2$, where $0 \leq a < \infty$ and $0 \leq b \leq \infty$. If $p \in M$ and z is a point at infinity (cf. Eberlein-O'Neill [4], which we give as a general reference for Hadamard manifolds), there exists a *horosphere* through p with center z. This is defined as follows: Denote the geodesic ray from p to z by γ, and consider the geodesic spheres through p with center $\gamma(t)$, $t > 0$. As t goes to infinity, these spheres converge to the horosphere. More precisely, the horospheres are the level surfaces of the *Busemann function* $F = \lim F_t$, where F_t is defined by $F_t(p) = d(p, \gamma(t)) - t$. In the flat case ($a = b = 0$), horospheres are just affine hyperplanes, and in the case of constant negative curvature, using the Poincaré model we see that horospheres are euclidean spheres internally tangent to the boundary sphere, minus the point of tangency. The main purpose of this paper is to show that, to a certain extent, the geometry of horospheres in M may be compared with that in the spaces of constant curvature $-a^2$ and $-b^2$, respectively. We give two examples:

1. (Theorem 4.6). *If $\mathscr{H}$ is a horosphere and h denotes the distance in $\mathscr{H}$ with respect to the induced metric, then for all $p, q \in \mathscr{H}$*

$$\frac{2}{a} \sinh \frac{a}{2} d(p, q) \leq h(p, q) \leq \frac{2}{b} \sinh \frac{b}{2} d(p, q) ,$$

where d is the distance function of M.

2. (Theorem 4.9). *If γ is a geodesic tangent to a horosphere $\mathscr{H}$, and if p, q are the projections of $\gamma(\pm\infty)$ onto $\mathscr{H}$, then*

$$\frac{2}{b} \leq h(p, q) \leq \frac{2}{a} .$$

Communicated by W. P. A. Klingenberg, September 2, 1975. This work was done under a program of the Sonderforschungsbereich "Theoretische Mathematik" at the University of Bonn. The first author was supported by the "Deutsche Forschungsgemeinschaft", and the second author by the "Schweizerischer Nationalfonds". The authors would like to thank P. Ehrlich and H. Karcher for many helpful discussions. In particular, we are obliged to H. Karcher for suggesting us a method to prove Theorem 4.6.