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GLOBAL PROPERTIES OF SPHERICAL CURVES

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Let α be a closed curve regularly embedded in Euclidean three-space satisfying suitable differentiability conditions. In addition, suppose α is nonsingular, i.e., free of multiple points. In 1968, B. Segre [4] proved the following about such curves.

Theorem. *If α is nonsingular and lies on a sphere, and* 0 *denotes any point of the convex hull of α with the condition that* 0 *(if lying on α) is not a vertex of α, then there are always at least four points of α whose osculating plane at each of those points passes through* 0. *If* 0 *is a vertex of α then there are at least three points of α whose osculating plane at each of those points passes through* 0.

All terms used in the statement of the theorem are defined later in this paper.

To quote H. W. Guggenheimer [2] who reviewed [4], "The 12-page proof is rather complicated." Here we present a shorter and hopefully more transparent proof of this theorem. In addition, we need only require that the spherical curve α be of class C^2 whereas Segre's proof requires α be of class C^3. Also, we obtain, with no extra effort, a similar theorem which holds if α's only singularity is one double point; in this case, the above mentioned minimums must be reduced by two.

In the last section of this paper we characterize spherical curves with the following property: for every point 0 of the convex hull of α, other than a vertex of α, there exists the same (necessarily even) number of distinct points of α whose osculating plane at each of those points passes through 0.

The proofs of many results in this paper ultimately depend on ideas contained in a paper by W. Fenchel [1].

Throughout this paper we use the following conventions. By a curve we mean a regular C^2 function $\alpha: D \to E^3$, where D is an interval (with or without end points) or a circle, and E^3 is Euclidean three-space. We let α denote both the function and its configuration $\alpha(D)$ in E^3. When D is a circle we say α is closed. If D is a closed interval we may sometimes refer to α as an arc. We say a point P in E^3 is a multiple point of α if it is the image of $k > 1$ points of D. If $k = 2$ then P is called a double point. At a multiple point P we will think of P as k distinct points each traversed once by α as we traverse D once. If α has no multiple points, then we say α is nonsingular.

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