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HOLOMORPHIC AND DIFFERENTIABLE TANGENT SPACES TO A COMPLEX ANALYTIC VARIETY

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An important invariant in the study of analytic varieties is the local embedding dimension. To measure this precisely one defines $T(V, \mathcal{O}_p)$, the tangent space to V at p with respect to the analytic functions. Similarly one can define tangent spaces with respect to the infinitely differentiable functions C^∞, and the k times continuous differentiable functions C^k, whose dimension is the local C^k or C^∞ embedding dimension. It is known [6], [18] that $T(V, C_p^\infty) = T(V, \mathcal{O}_p)$. In this paper we strengthen that result as follows: there is a locally bounded function $k\colon V \to \boldsymbol{Z}^+$ such that $T(V, C_p^{k(p)}) = T(V, \mathcal{O}_p)$.

An outline of the paper is the following. First show that for curves, k can be picked $\leq N$, where N is the exponent of the conductor. Then find a curve C in V such that $T(C, \mathcal{O}_p) = T(V, \mathcal{O}_p)$. The local boundedness of k follows by showing there is an upper bound for the conductor number of all nearby linear one-dimensional sections of V. One finds this upper bound by stratifying V into finitely many ''equisingular'' varieties so that the conductor number is constant on each one.

For curves, we derive some precise estimates for k, and in § 3 we give examples to show these estimates are in general the best possible. Also for each k we show there exists a variety V so that $T(V, C^{k-1}) \neq T(V, \mathcal{O})$, but $T(V, C^k) = T(V, \mathcal{O})$, that is, k is the precise critical degree of differentiability. This enables us to construct a Stein complex space X with no C^∞ embedding in any $\boldsymbol{C}^m$, but for every k there is a C^k embedding into some $\boldsymbol{C}^n$.

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1. Definitions and preliminaries

From [18] we have all of the following. Let V be a complex analytic variety in $\boldsymbol{C}^n$, $p \in V$, C_p^k the ring of germs at p of k times continuously differentiable

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