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A GENERAL APPROACH TO MORSE THEORY

A. J. TROMBA

The Morse theory of critical points was extended by Palais and Smale [10], [16] to a certain class of functions on Hilbert manifolds. However, there are many variational problems in a nonlinear setting which for technical reasons are posed not on Hilbert but on Banach manifolds of mappings. For example, the Plateau problem, the existence of harmonic mappings between finite dimensional Riemannian manifolds, and the fixed endpoint solution to the Euler equations of hydrodynamics to name a few. It would therefore be desirable to have an infinite dimensional Morse theory which applies to these problems. The purpose of this paper is to extend Morse theory to manifolds modelled on Banach spaces and to show how this theory applies to the problem of geodesics on finite dimensional Riemannian manifolds. Other applications will be given in future papers.

Such extensions have already been given by Uhlenbeck [22], [23] and we build upon her work to some extent. Our theory has the advantages (a) that the definition we give of nondegenerate critical point (§ 2) is intrinsic, that is, does not depend on the choice of a particular coordinate neighborhood, and (b) we abandon the condition (C) of Palais and Smale and replace it with a condition which works in a much more general setting (see the discussion at the end of § 1 and the beginning of § 3). In addition this new theory fits nicely with the authors [15], [20] generalization of vector field index theory to the Banach manifold category. Finally we assume that the mappings f which we consider are of class C^2. This is in the spirit of Smale's approach to Morse theory [6].

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For condition (C) to be satisfied Palais needed the manifold of L_1^2 maps of the interval into V. We show that in our theory we are free to choose *any* Sobolev manifold of maps functor L_k^p, $k > 0$. Condition (C) is then violated but not our conditions. The notion of nondegeneracy does not depend on the model space.

1. Preliminaries and a review of standard theory

Let M be a C^k, $k > 1$ Banach manifold and let TM denote its tangent bundle

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