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THE HOMOLOGY THEORY OF THE CLOSED GEODESIC PROBLEM

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The problem—*does every closed Riemannian manifold of dimension greater than one have infinitely many geometrically distinct periodic geodesics*—has received much attention. An affirmative answer is easy to obtain if the fundamental group of the manifold has infinitely many conjugacy classes, (up to powers). One merely needs to look at the closed curve representations with minimal length.

In this paper we consider the opposite case of a *finite* fundamental group and show by using algebraic calculations and the work of Gromoll-Meyer [2] that *the answer is again affirmative if the real cohomology ring of the manifold or any of its covers requires at least two generators.*

The calculations are based on a method of the second author sketched in [6], and he wishes to acknowledge the motivation provided by conversations with Gromoll in 1967 who pointed out the surprising fact that the available techniques of algebraic topology loop spaces, spectral sequences, and so forth seemed inadequate to handle the "rational problem" of calculating the Betti numbers of the space of all closed curves on a manifold. He also acknowledges the help given by John Morgan in understanding what goes on here.

The Gromoll-Meyer theorem which uses nongeneric Morse theory asserts the following. Let $\Lambda(M)$ denote the space of all maps of the circle S^1 into M (not based). Then there are *infinitely many* geometrically distinct periodic geodesics in *any* metric on M *if* the Betti numbers of $\Lambda(M)$ are unbounded. (The round 2-sphere shows the condition is not actually necessary.)

In [6] a description of the *minimal model of* $\Lambda(M)$ is given in terms of the minimal model of M, and is valid if, for instance, M is simply connected. This gives an explicit algorithm for calculating the Betti numbers of $\Lambda(M)$. This algorithm yields the fact that the Betti numbers of $\Lambda(M)$ are always nonzero in an infinite number of dimensions (which can be taken to lie in an arithmetic sequence.)

We explicate and extend this study of $\Lambda(M)$ here by

(i) giving the description and proof of the algorithm for the homology when $\pi_1 M = \{e\}$, and by

(ii) showing that the Betti numbers of $\Lambda(M)$ are unbounded if and only if

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