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ALMOST REGULAR CONTACT MANIFOLDS

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If M^{2n+1} is a C^∞-manifold such that a 1-form ω of class C^∞ is defined over M with the property that $\omega \wedge (d\omega)^n = \omega \wedge d\omega \wedge \cdots \wedge d\omega \neq 0$, then M is said to be a contact manifold and ω a contact form. Classical examples are provided by the $(2n + 1)$-sphere S^{2n+1} and the total space of the cotangent sphere bundle of any $(n + 1)$-dimensional manifold X. Indeed the latter arises as a level of constant energy in the momentum phase space of a Hamiltonian system, see for example [12], and historically has provided the incentive for the study of contact structures in general. In 1958 W. M. Boothby and H. C. Wang introduced the notion of a *regular* contact form; this, loosely speaking, is such that every point x of M has a neighborhood U which is pierced exactly once by any integral curve of the vector field Z dual to ω, [1]. Regularity is equivalent to the existence of a free S^1-action on M, whose orbit space is a symplectic manifold with integral fundamental class. In this paper we weaken the definition of regularity to allow an integral curve to pierce U a finite number of times, and prove that this implies the existence of a C^∞ S^1-action on M without fixed points, and with only finitely many isotropy groups. The manifold M can be fibred in the sense of Seifert, and assuming the M/S^1 is a C^∞-manifold, it is easy to see that the quotient space of principal orbits admits an integral symplectic form. But perhaps the more interesting result is the converse (Theorem 3 below) which allows us to construct an almost regular contact form on the total space of a suitable Seifert fibration. We apply this in particular, when the base is a projective algebraic variety, and obtain examples of contact forms on $(n - 1)$-connected $(2n + 1)$-manifolds, $n = 2$ or odd.

The existence of a contact form ω imposes restrictions on the tangent bundle $\tau(M)$. On $\boldsymbol{R}^{2n+1}$ the form $\omega = dz + x_1dx_2 + \cdots + x_{2n-1}dx_{2n}$ is contact, and by Darboux' Theorem [8, p. 132], every point x of the contact manifold M has a coordinate neighborhood on which ω can be expressed in this way. It follows that M has an atlas for which the coordinate transformations are compatible with this standard contact form, and hence that the structural group of $\tau(M)$ reduces to $U(n) \oplus 1$, at least when M is orientable [3, 2.3.2]. This provides the definition of an almost contact manifold, and an almost structure is integrable if it is induced by a contact form ω on M. On an *open* manifold

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