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# TAUT IMMERSIONS OF NONCOMPACT SURFACES INTO A EUCLIDEAN 3-SPACE

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Let $f$ be a smooth ($C^\infty$) immersion of a smooth surface $M$ into a Euclidean $n$-space $R^n$. For $p \in R^n$, $x \in M$, the function $L_p(x)$ is defined by $L_p(x) = d(f(x), p)^2$, where $d$ is the Euclidean distance in $R^n$.

The immersion $f$ is said to be proper if the inverse image under $f$ of every compact subset of $R^n$ is compact. Following the terminology of Carter and West [2] the immersion in said to be taut, if $f$ is proper and every Morse function $L_p$, $p \in R^n$, has the minimum number of critical points required by the Morse inequalities [12, p. 270]. This definition is valid for noncompact as well as compact surfaces $M$.

For $M$ compact and connected, an immersion $f: M \to R^n$ is taut if and only if $f(M)$ has the spherical 2-piece property, that is, no hyperplane or hypersphere in $R^n$ divides $f(M)$ into more than two pieces (see [1] for more detail).

Primarily through the work of Kuiper [10] and Banchoff [1], all taut immersions of compact surfaces into $R^n$ are known (see [2, p. 711] for a complete listing). In particular, Banchoff proved by considering the spherical 2-piece property that the image of a taut immersion of a compact connected surface in $R^3$ must be either a Euclidean sphere or a cyclide of Dupin.

The cyclides of Dupin will be discussed in detail in § 1. For now it will suffice to say that a compact cyclide is either a torus of revolution or a surface obtained by inverting such a torus in a sphere whose center is now on the torus. A complete noncompact cyclide is either a circular cylinder or a surface obtained by inverting a torus of revolution in a sphere whose center is on the torus.

The result of Banchoff was generalized by Nomizu and Rodriguez [13] who showed that a taut immersion of an $m$-sphere in $R^n$ must, in fact, be a Euclidean sphere $S^m \subset R^{m+1} \subset R^n$. Several other generalizations were obtained by Carter and West [2]. The author has also found characterizations of certain submanifolds of hyperbolic space [3] and complex projective space [4] in terms of the distance functions in those spaces.

In this paper, we are concerned with taut immersions of noncompact surfaces. Carter and West [2, p. 710] have proven that if $f: M \to R^n$ is a taut immersion of a noncompact surface, then $f(M) \subset R^4 \subset R^n$. If $f(M)$ is not actually contained in $R^3$, then $f(M) = P(V)$, where $V$ is a Veronese surface

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