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NATURAL TENSORS ON RIEMANNIAN MANIFOLDS

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Suppose that for each differentiable manifold M (without boundary) and each C^∞ metric g, we are given a C^∞ tensor field $t_{(M,g)}$ satisfying the following naturality axiom:

If $\varphi\colon (M,g) \to (N,h)$ is an isometry of M onto an open subset of N, then $\varphi_* t_{(M,g)} = t_{(N,h)} \mid \varphi M$.

In these circumstances we say that t is a *natural tensor*.

Our objective is to elucidate the nature of natural tensors. It will emerge that the situation is too complicated for there to be any hope of a complete classification. Therefore we try to find additional conditions which can be imposed on a natural tensor which will imply that it lies in a good class of natural tensors.

Our main results are as follows. In § 5 we classify all natural tensors which depend in a polynomial way of the ∞-jet of g. In § 6 we show that it is sufficient for the dependence on the ∞-jet to be a differentiable dependence (we demand C^∞-dependence in Theorem 6.2, but the proof obviously goes through with less differentiability), if in addition the tensor is homogeneous (Definition 5.1): these two conditions imply polynomial dependence. In Theorem 7.3 we prove a special result where only homogeneity is assumed and nothing whatever concerning the dependence on the ∞-jet of g. The fact that this is not trivial is shown by the existence of an example of a natural tensor depending only on the 4-jet of g, but with the dependence not even continuous (Theorem 4.1). We observe in passing that the space of germs of C^∞ Riemannian manifolds of dimension 2 can be parametrized in terms of the orbits of a linear 0(2)-action on an infinite dimensional vector space (Corollary 2.4). Finally we show that there is a unique "natural" connection ∇ for Riemannian manifolds-namely the Levi-Civita connection. In other words the fact that ∇ is torsion free and preserves the metric follows from the naturality. (For a precise statement, see Theorem 5.6.)

This works was stimulated by G. Lusztig when he asked whether the Levi-Civita connection was the unique natural connection. This was during lectures on the work of Atiyah, Bott and Patodi [1] whose treatment of Gilkey's theorem has heavily influenced this paper. In fact Gilkey's theorem deals with the problem of classifying natural q-forms. Here we relax the condition that

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