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MAPPINGS OF BOUNDED DILATATION OF RIEMANNIAN MANIFOLDS

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1. Introduction

Let M and N be Riemannian manifolds of dimensions m and n, respectively. Recently, two of the authors introduced the concept of a quasiconformal mapping $f: M \to N$ and applied it to obtain distance and (intermediate) volume decreasing properties of harmonic mappings between Riemannian manifolds of different dimensions [2], [3]. In this paper the concept of a mapping $f: M \to N$ of bounded dilatation is introduced which is more general and natural than that of a K-quasiconformal mapping when m and n are greater than 2. An example of such a mapping which is not K-quasiconformal is given which is even harmonic. In § 5, generalizations of the Schwarz-Ahlfors lemma as well as Liouville's theorem and the little Picard theorem are given for this class of mappings.

Let $f: M \to N$ be a harmonic mapping of bounded dilatation of Riemannian manifolds. If the upper bound $\|f_*\|^2$ of the ratio of distances attains a maximum at $x \in M$, then under suitable conditions on the bounds of the sectional curvatures at x and $f(x)$, f is distance decreasing.

If M is a complete connected Riemannian manifold of constant negative curvature $-A$, in particular, if M is the unit open m-ball with the hyperbolic metric of constant curvature $-A$, then the condition on $\|f_*\|$ may be dropped by virtue of the technique employed in § 5. Indeed, let N be a Riemannian manifold with sectional curvatures bounded above by a negative constant depending on A. Then, if $f: M \to N$ is a harmonic mapping of bounded dilatation, it is distance decreasing.

The technique employed to prove this statement also yields the following fact.

Let M be a complete connected locally flat Riemannian manifold and let N be an n-dimensional Riemannian manifold with negative sectional curvature bounded away from zero. Then, if $f: M \to N$ is a harmonic mapping of bounded dilatation, it is a constant mapping.

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