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THE SPECTRAL GEOMETRY OF A RIEMANNIAN MANIFOLD

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Introduction

Let M be a compact smooth d-dimensional Riemannian manifold without boundary. Let $X = (X_1, \cdots, X_d)$ be a system of local coordinates centred at x_0. The metric tensor is given by

$$ds^2 = g_{ij} dX_i \otimes dX_j \qquad (\text{summed over } i, j = 1, \cdots, d) \ .$$

We adopt the convention of summing over repeated indices except where otherwise indicated. Let (g^{ij}) denote the inverse of the matrix (g_{ij}).

Let V be a smooth vector bundle over M and let D be a second order differential operator on V. Let $e = (e_1, \cdots, e_r)$ be a local frame for V defined near x_0. The coordinate system and frame e comprise a local system which identifies a neighborhood of M with R^d and a portion of V with $R^d \times R^r$. Using this local system, we express the operator D:

$$D = -\left(h^{ij}\frac{d^2}{dx_i dx_j} + a_i \frac{d}{dx_i} + b\right) ,$$

where h^{ij}, a_i, and b are square $r \times r$ matrices. Let $\xi \in T^*M$ and define

$$a^2(x, \xi) = h^{ij}\xi_i\xi_j \ , \quad a^1(x, \xi) = -ia_i\xi_i \ , \quad a^0(x, \xi) = -b \ .$$

The leading order symbol of D is a^2, which is defined invariantly. The lower order terms depend upon the local system chosen.

For the rest of this paper, we assume that the leading symbol is given by the metric tensor, i.e., that $h^{ij} = g^{ij}I = g^{ij}$, which implies $a^2(x, \xi) = |\xi|^2$. We omit multiplication by the identity matrix on V, and apply the functional calculus to define the operator $\exp(-tD)$ for $t > 0$. $\operatorname{Exp}(-tD)$ is an infinitely smoothing operator from $L^2(V) \to C^\infty(V)$. It is defined by a kernel function $K(t, x, y, D)$ such that:

$$\exp(-tD)u(x) = \int K(t, x, y, D)u(y) d\operatorname{vol}(y) \ ,$$

$K(t, x, y, D)$ maps V_y to V_x, $d\operatorname{vol}(y)$ is the Riemannian measure. Seeley [8] proved that $K(t, x, x, D)$ has an asymptotic expansion as $t \to 0^+$ of the form:

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