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EXAMPLES OF NONVANISHING CHERN-SIMONS INVARIANTS

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Introduction

In this paper we study the Chern-Simons invariants. These invariants of $(4n-1)$-dimensional Riemannian manifolds first appeared in Chern-Simons [1]. They are obstructions to conformal immersion of the Riemannian manifold in Euclidean space in much the same way as the Pontrjagin classes are to topological immersion. In Chern-Simons [1] a 3-dimensional example was given whose Chern-Simons invariant was nonzero. However, no higher-dimensional examples were given. Our first theorem gives a simple algebraic formula for these invariants for a spherical space form. In particular, for the Lens spaces $L(p; q_1, q_2, \cdots, q_{2n})$ the invariants are expressible in terms of the elementary symmetric functions of $q_1, q_2, \cdots, q_{2n}$ modulo p. Using this and judiciously choosing p and the q_i's one can produce for each n, infinitely many Lens spaces $L(p; q_1, q_2, \cdots, q_{2n})$ which immerse smoothly in R^{4n} but not conformally in $R^{4n+2n-2}$. This is the "best-possible" non-immersion result obtainable with the Chern-Simons invariants. For example, the 15-dimsional Lens space $L(137; 1, 10, 100, 41, 136, 127, 37, 96)$ immerses smoothly in R^{16} but not conformally in R^{22}. As another application of our calculation for Lens spaces we give a residue formula for the Pontrjagin numbers of a $4n$-manifold admitting a periodic diffeomorphism of prime order. We give here the formula for the case where the diffeomorphism f has only isolated fixed points. Let $Q(p_1, p_2, \cdots, p_n)$ be a polynomial of the right weight in the Pontrjagin classes p_i to obtain a Pontrjagin number $Q(M)$. Let $m_1, m_2, \cdots, m_k$ be the fixed points of f. If p is the order of f, then $Q(M) \equiv \sum_{i=1}^{k} \operatorname{Res}(f, m_i)$, modulo p, where $\operatorname{Res}(f, m_i)$ is calculated as follows. Since f leaves m_i fixed, df maps the tangent space of M at m_i to itself. One can always (by averaging) assume f preserves a metric on M, so $df(m_i)$ is a rotation of order p. Let $\theta_1, \theta_2, \cdots, \theta_{2n}$ be its rotation angles; that is, $df(m_i)$ is similar to a block matrix:

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