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THE DE RHAM COHOMOLOGY OF SUBCARTESIAN SPACES

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The notion of differentiable subcartesian space is a generalization of that of differentiable manifold. Arbitrary subsets of $\boldsymbol{R}^n$ are special examples as well as differentiable manifolds with boundary, or corners, and analytic or semi-analytic spaces. In [7] we constructed the category of C^∞-subcartesian spaces and introduced the calculus of tensor fields and differential forms. In this sequel to [7] we study the cohomology algebra formed from those differential forms.

In § 1 we define the de Rham cohomology of a C^∞-subcartesian space. In § 2 we establish the Eilenberg-Steenrod axioms on an appropriate admissible category of pairs of subcartesian spaces. In § 3 we show by example that the de Rham and Čech cohomologies are distinct. We then establish a spectral sequence which has its E_2-terms in sheaf cohomology and which converges in the de Rham cohomology. We introduce a graded-sheaf invariant $\mathscr{H}(S)$ of a differentiable subcartesian space S, the *de Rham sheaf of S*, whose vanishing in higher degrees is sufficient for the de Rham cohomology to be naturally isomorphic to the sheaf cohomology with coefficients in $\mathscr{H}^0(S)$. If S is locally contractible, then $\mathscr{H}^0(S) = \boldsymbol{R}$ and $\mathscr{H}^k(S) = 0$ for $k > 0$, thus giving a natural isomorphism of the de Rham and sheaf- theoretic cohomology theories. We finish with an appendix on the C^k-cohomology, showing that it is not a topological invariant.

It is perhaps worth while to compare the cohomology theory developed here with those of [10], [11], and [12]. In [10] Schwartz constructed a cohomology theory which coincides with Čech cohomology on finite dimensonal compact spaces. Example 3.1 shows that this is not always the case for our theory. In [11] Smith constructed an exterior differential algebra and cohomology theory for each pair $(X, \mathscr{F})$, where X is a topological space and $\mathscr{F}$ is a set of continuous $\boldsymbol{R}$-valued functions on X. One might expect our theory to follow as a special case of Smith's when X is a C^∞-subcartesian space and $\mathscr{F} = C^\infty(X)$, but Example 3.15 shows that this is not the case. In [12] Spallek considered several notions of differential forms on differentiable spaces and stated a de Rham isomorphism theorem. In [7] we showed that the differential forms as defined for subcartesian spaces and the differential forms of [12] are different. Whether

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