J. DIFFERENTIAL GEOMETRY
10 (1975) 551-574

CALCULUS ON SUBCARTESIAN SPACES

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Introduction

The notion of a *differentiable subcartesian space* is a generalization of that of a differentiable manifold and includes *arbitrary* subsets of $\boldsymbol{R}^n$ as special examples. In this paper we construct the category of differentiable subcartesian spaces and develop the calculus of differentiable mappings, vector and tensor fields, and exterior differential forms.

More precisely, a differentiable subcartesian space of class C^k is a Hausdorff space equipped with an atlas of local homeomorphisms into various cartesian spaces $\boldsymbol{R}^n$, each pair of which satisfies a condition of C^k-compatibility similar to that satisfied by charts of a C^k-manifold. For the sake of simplicity we shall restrict attention to the C^∞-case. The necessary modifications for other smoothness categories are obvious for the most part, although the C^k-theory, for instance, is not without independent interest (see [4]). The calculus of differential forms gives rise to the de Rham cohomology of a subcartesian space, and we shall introduce that theory in a sequel to this paper.

In brief outline, our results are the following. The category of C^∞-subcartesian spaces possesses a tangent functor T sending each S into its tangent pseudo-bundle TS and each differentiable mapping f into the corresponding induced mapping f_*. As one would guess from the terminology, TS is not a vector bundle but is a fiber space, the dimension of whose fibers may vary on connected components of S. We introduce the notion of *differentiable vector pseudo-bundle* and show how tensor products and other covariant differentiable functors on the category of real vector spaces may be naturally lifted to vector pseudo-bundles. Thus having the "contravariant" tensors and their fields, we introduce the modules of covariant and mixed tensor fields and determine their dual modules. We then introduce the Lie module of Lie derivatives.

In each case the idea is to lift classical objects and constructions (e.g., vector fields and their Lie products) from the ambient cartesian spaces up to S via charts. As in the special case of differentiable manifolds, this method always requires one to check invariance under coordinate changes, but there are now two more things to be checked. The constructions made in $\boldsymbol{R}^n$ with local representatives of objects on S must not depend on the choices of these local

Communicated by S. S. Chern, April 1, 1974. Research partially supported by the National Science Foundation Grant GP-16292.