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GEODESIC FOLIATIONS BY CIRCLES

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1. Introduction

Smooth foliations by circles of compact three-manifolds have been completely analysed by D. B. A. Epstein in the paper [2]. Essentially, he shows that all such foliations arise as a decomposition of the manifold by the orbits of a smooth circle action. The theorem of this paper shows that the same is true of an arbitary smooth manifold, compact or not, with a foliation by circles satisfying a certain (rather strong) regularity condition.

It is known that not all foliations by circles arise as the orbits of some action by S^1; indeed, the paper [2] presents a foliated noncompact three-manifold as a counter-example to such a proposition. However, it is an open question whether or not such examples exist in the case of a foliated *compact* manifold of dimension greater than three.

A C^r flow on a C^r manifold M is a C^r action $\mu: \boldsymbol{R} \times M \to M$ of the additive reals on M. A C^r flow without fixed points, each of whose orbits is compact, gives rise to a C^r foliation of the manifold by circles. Further, any C^r foliation by circles of a manifold M gives rise to a C^r flow on (a double cover of) M. The version of the theorem presented here is stated for flows; an equivalent version for circle foliations in terms of differential forms is readily obtainable (see § 2). The theorem is the following.

Theorem. *Let $\mu: \boldsymbol{R} \times M \to M$ be a C^r action $(3 \le r \le \infty)$ of the additive group of real numbers with every orbit a circle, and M a C^r manifold. Then there is a C^r action $\rho: S^1 \times M \to M$ with the same orbits as μ if and only if there exists some riemannian metric on M with respect to which the orbits of μ are embedded as totally geodesic submanifolds of M.*

Finding some such metric given a circle action on M is easy (see § 3); the proof of the converse requires a little more effort. The author wishes to thank David Epstein for his gentle encouragement and for his many helpful suggestions.

2. The invariant one-form

Suppose a riemannian metric exists on the manifold M as in the theorem. At each point $m \in M$ choose a unit vector T_m in the direction of the flow μ.

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