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A GENERALIZATION OF THE CURVATURE INVARIANT

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Introduction

Let M be a manifold, and D_θ a linear connection on $T(M)$. Classically it has been shown that all local parallel vector fields, i.e., all vector fields X with $D_\theta X = 0$, must satisfy $X \in \bigcap_l \ker \nabla^l R$ where the Rimannian curvature tensor R and its covariant derivatives $\nabla^l R$ are regarded as linear maps $\nabla^l R : T(M) \to \bigotimes_{l+2} T^*(M) \otimes T(M)$. See, e.g., [1].

A related problem is the following: when is a tensor of type [1, 1] on a Riemannian or affinely connected manifold the covariant derivative of a vector field?

If E is a vector bundle and D_θ a connection on E, then the analogous questions can be asked. We study both questions in this general setting. In this context the maps $\nabla^l R$ no longer exist. However, we introduce a set of invariants $\Theta^{(l)}$ which are called higher order curvatures and serve the same purpose in this context. The definition is completely general; we associate to any differential operator $\underline{E} \xrightarrow{D} \underline{F}$ a sequence of $\mathcal{O}$-linear maps $\Theta^{(l)}(D) : E \to G_{l,0}$ where $G_{l,0}$ is canonically defined. (E and F are vector bundles.) In the present context $\Theta^{(1)}(D_\theta) = \Theta$ is the classical curvature. It is not true, however, that $\Theta^{(l)} = \nabla^{l-1}\Theta$ when $E = T(M)$, but they do have a close relationship as we shall see. Moreover, in the appropriate sense the $\Theta^{(l)}(D_\theta)$ are covariant derivatives of Θ and obey Bianchi-type identities.

The $\Theta^{(l)}$ also play a role in the study of the nonhomogeneous equation $D_\theta f = \alpha$. In fact when D_θ has constant rank, it is shown that if $E' = \bigcap_l \ker \Theta^{(l)}$, D_θ restricts to a flat connection on E'. This allows us to reduce the study of $H(M, \omega)$, where ω is the sheaf of germs of local solutions of $D_\theta f = 0$, to the case where D_θ is flat. Our preliminary calculation of $H(M, \omega)$ yields satisfactory results in two cases.

a) When the base manifold M of E is simply connected.

b) When M is a Riemannian manifold of strictly positive or strictly negative sectional curvature, and D_θ is the Riemannian connection on $T(M)$.

Some of the results of this paper were announced in [3].

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