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A THEOREM OF GÉOMÉTRIE FINIE

WILLIAM F. POHL

1. Introduction

In a paper in the Kodaira *Festschrift* R. Thom [6] gave a proof, admittedly incomplete, of the following. Let $V^{2k} \subset P_{n+k}$ be a real compact embedded submanifold of real dimension $2k$ of a complex projective space of complex dimension $n+k$. Suppose there exists an everywhere dense subset $U \subset G_{n,k}$ of the Grassmannian of all complex projective subspaces of complex dimension n of P_{n+k}, such that if $u \in U$ then $u \cap V^{2k}$ consists of exactly m points, where m is independent of u. Then V^{2k} is an algebraic subvariety of P_{n+k}; the flat case is excluded. In this paper we give a complete and corrected statement and proof of this result. Moreover, we will allow V^{2k} to have certain singularities.

By a *semi-real flat L* we mean the closure in P_{n+k} of an affine subspace $L_0 \subset R^{2(n+k)}$, where we make the canonical identification $R^{2(n+k)} = C^{n+k} \subset P_{n+k}$. These may be classified in the following way. Apply a complex affine transformation (which is also a real affine transformation) to make L_0 pass through the origin. Let $I\colon C^{n+k} \to C^{n+k}$ be the multiplication by $i = \sqrt{-1}$. We call the real dimension of $L_0 \cap I(L_0)$ the *type* of L. If a complex projective transformation sends L into a semi-real flat, then it preserves the type, so that a semi-real flat is classified up to a complex projective transformation by its dimension j and its type t, where $0 \leq t \leq j$ with every such even t possible. If $t = j$, then L is a complex projective subspace, and if $t = 0$ then L is complex projectively equivalent to the real projective space P^j with its canonical embedding $P^j \subset P_j \subset P_{n+k}$. In the intermediate cases $0 < t < j$, L is singular; in fact it is a kind of cone.

We say that a continuous map $f\colon X \to Y$ of topological spaces is *proper onto its image* if for every compact subset $A \subset f(X)$, $f^{-1}(A)$ is compact. We now state the main result.

Theorem. *Let $V \subset P_{n+k}$ be a compact subset. Suppose there exists a closed subset $S \subset V$ such that the closure of $V - S$ is all of V, and an immersion $f\colon M \to P_{n+k}$ of class C^4 of a differentiable manifold M of (real) dimension $2k$ which maps M onto $V - S$ and which is proper onto its image. Suppose further:*

1) *there exists an everywhere dense subset $T \subset G_{n+1,k-1}$ such that if $v \in T$*

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