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THE STRUCTURE OF δ-PINCHED MANIFOLDS WITH THE FUNDAMENTAL GROUP $\pi_1(M) = Z_3$

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The present paper is a continuation of the differentiable pinching theorems for the sphere (see [7]), and the real (see [3] and [4]) and the complex (see [5]) projective spaces. The diffeotopy theorem plays an essential role for obtaining the dimension independency in the proof of the sphere pinching theorem. In order to make a fibre preserving diffeotopy between the Hopf fibration $S^{2n+1} \to S^{2n+1}/S^1 = P(C)^n$ and the free S^1 action on S^{2n+1} which is caused by a Riemannian manifold N with certain conditions (see [5]), we made heavy use of the strong diffeotopy theorem to get the diffeomorphism between the complex projective space and such an N. In the real projective pinching theorem, a fibre preserving diffeotopy between the antipodal map on S^m and the involutive diffeomorphism on S^m obtained from a δ-pinched M with $\pi_1(M) = Z_2$ is constructed easily by the diffeotopy theorem, and in this case we again obtain the dimension independency. The reason why we need not use the strong diffeotopy theorem in the real projective pinching is based on the following two facts. First, for each point p on a δ-pinched M with $\pi_1(M) = Z_2$, the cut locus $C(p)$ of p is a compact hypersurface of M without boundary. Second, the deck transformation on the universal covering Riemannian manifold $\tilde{M}$ of M leaves the inverse image $\pi^{-1}(C(p))$ of $C(p)$ invariant, where $\pi: \tilde{M} \to M$ is the covering projection.

However, if the order of $\pi_1(M)$ is greater than 2, it will not be easy to investigate the structure of cut locus $C(x)$ of a point x on M. This is because $C(x)$ has nonempty boundary, and furthermore each element of the deck transformation group does not leave $\pi^{-1}(C(x))$ invariant. For instance, let $L^{2n+1}(r_1, \cdots, r_n; k) = S^{2n+1}/G$ be a general lens space of constant curvature 1 of the type $(r_1, \cdots, r_n; k)$, i.e., G has the generator g such that it is expressed in terms of the orthonormal basis $(e_1, \cdots, e_{2n+2})$ of R^{2n+2} as follows:

$$g = \begin{bmatrix} R(r_1/k) & & & \\ & R(r_2/k) & & \\ & & \ddots & \\ & & & R(r_n/k) \end{bmatrix}, \qquad R(\alpha) := \begin{bmatrix} \cos 2\pi\alpha & \sin 2\pi\alpha \\ -\sin 2\pi\alpha & \cos 2\pi\alpha \end{bmatrix}.$$

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