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THE STRUCTURE OF COMPACT RICCI-FLAT RIEMANNIAN MANIFOLDS

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0. Introduction and preliminaries

An interesting problem in riemannian geometry is to determine the structure of complete riemannian manifolds with Ricci tensor zero (Ricci-flat). In particular one asks whether such manifolds are flat. Here we show that any compact connected Ricci-flat n-manifold M^n has the expression

$$M^n = \Psi \backslash T^k \times M^{n-k} ,$$

where k is the first Betti number $b_1(M^n)$, T^k is a flat riemannian k-torus, M^{n-k} is a compact connected Ricci-flat $(n - k)$-manifold, and Ψ is a finite group of fixed point free isometries of $T^k \times M^{n-k}$ of a certain sort (Theorem 4.1). This extends Calabi's result on the structure of compact euclidean space forms ([7]; see [20, p. 125]) from flat manifolds to Ricci-flat manifolds. We use it to essentially reduce the problem of the construction of all compact Ricci-flat riemannian n-manifolds to the construction in dimensions $< n$ and in dimension n to the case of manifolds with $b_1 = 0$ (see § 4). We also use it to prove (Corollary 4.3) that any compact connected Ricci-flat manifold M has a finite normal riemannian covering $T \times N \to M$ where T is a flat riemannian torus, $\dim T \geq b_1(M)$, and N is a compact connected simply connected Ricci-flat riemannian manifold. This extends one of the Bieberbach theorems [4], [20, Theorem 3.3.1] from flat manifolds to Ricci-flat manifolds, and reduces the question of whether compact Ricci-flat manifolds are flat to the simply connected case. J. Cheeger and D. Gromoll have pointed out to us that this extension also follows from their proof of [8, Theorem 6]. Our direct proof however uses considerably less machinery than their deeper considerations of manifolds of nonnegative curvature.

As a consequence of these results, we can give a variety of sufficient topological conditions for Ricci-flat riemannian n-manifolds M to be flat. For example, if the homotopy groups $\pi_k(M) = 0$ for $k > 1$, or the universal covering of M is acyclic (Theorem 4.6), or M has a finite topological covering by a

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