J. DIFFERENTIAL GEOMETRY
10 (1975) 239-252

COMPACT QUOTIENT SPACES OF C^2 BY AFFINE TRANSFORMATION GROUPS

TATSUO SUWA

The purpose of this paper is to classify the compact complex surfaces of the form C^2/G, where G is a properly discontinuous and fixed point free group of affine transformations of the two-dimensional complex vector space C^2. Except for the use of some theorems on numerical characters of a compact complex surface, the method is mostly elementary.

§ 1 contains preliminary considerations on some properties of a fixed point free affine transformation group of C^2. In § 2 we perform the classification. Denoting by b_1 the first Betti number of the quotient space $S = C^2/G$, we prove that if $b_1 = 4$ then S is a complex torus (Theorem 1), if $b_1 = 3$ then S is a fiber bundle of elliptic curves over an elliptic curve (Theorem 2), if $b_1 = 2$ then S is a hyperelliptic surface (Theorem 3), and if $b_1 = 1$ then S is an elliptic surface over the projective line with multiple singular fibers (Theorem 4).

1. A fundamental lemma

Let G denote a group of affine transformations of the two-dimensional complex vector space C^2. Assume the action of G is (A) properly discontinuous, i.e., for any pair (K_1, K_2) of compact subsets in C^2, the set $\{g \in G \mid gK_1 \cap K_2 \neq \emptyset\}$ is finite, and (B) fixed point free, i.e., for all $g \in G$, $g \neq 1$, g has no fixed points. Thus the quotient space C^2/G is a complex manifold of complex dimension 2. Finally we assume (C) C^2/G is compact. The problem is to classify the compact complex surfaces of the form C^2/G. In this section we prove a fundamental lemma for this purpose.

First of all, each element g of G is expressed by a 3×3 matrix:

$$g = \begin{pmatrix} a_{11}(g) & a_{12}(g) & b_1(g) \\ a_{21}(g) & a_{22}(g) & b_2(g) \\ 0 & 0 & 1 \end{pmatrix},$$

which acts on $C^2 = \{z \mid z = (z_1, z_2)\}$ by

Received June 28, 1972, and, in revised form, April 2, 1974. Research partially supported by an NSF grant. After this paper was submitted, the author was informed that the material in §1 and some of the preliminary material in §2 had been published by J. P. Fillmore and J. Scheuneman, *Fundamental groups of compact complete locally affine complex surfaces*, Pacific J. Math. **44** (1973) 487–496.