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SMOOTHNESS OF HOROCYCLE FOLIATIONS

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1. Introduction

Let SM denote the unit tangent bundle of a compact C^∞ Riemannian manifold M. Suppose that M has everywhere negative sectional curvature. In [1] Anosov proved that the geodesic flow φ on SM is of a certain type, called "Anosov" by later writers, and defined below.

Associated with any Anosov flow φ is a foliation by "strong stable manifolds"; this is called the *horocycle foliation* in the special case where φ is the geodesic flow on SM and M has negative curvature. The strong unstable manifolds provide another isomorphic horocycle foliation.

The *leaves* of these foliations are as smooth as the Anosov flow φ, but Anosov showed that the *foliations* are not in general of class C^1, even when φ is real analytic.[1] However, when M has dimension two or the curvature is $\frac{1}{4}$-pinched, we shall prove that the horocycle foliations (and even their tangent plane fields) *are* of class C^1. In the course of the proof, the fact that "negative curvature $\Rightarrow$ Anosov geodesic flow" falls out naturally. Our methods in §§ 5, 6 resemble those of Anosov and Sinai [2].

This smoothness result was suggested to us by an analogous situation we encountered in [8]; there, we showed that the strong stable manifold foliation of an Anosov diffeomorphism f is of class C^1 provided that either the strong stable manifolds have codimension one in M or the spectrum of Tf is "bunched". These cases are analogous to (i), (ii) below.

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2. The smoothness theorem

Let M be a C^∞ compact boundaryless manifold with a C^∞ Riemann structure $\mathscr{R}$. The geodesics of $\mathscr{R}$ give rise to the geodesic flow φ on the tangent bundle TM of M:

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[1] It is amusing that, to mean "generic", Russian mathematicians, such as Anosov, use a word translated from Russian to English as "rough". Here is an example where roughness is likely to *be* generic.