J. DIFFERENTIAL GEOMETRY
10 (1975) 207-219

SOME REMARKS ON STABILITY OF FOLIATIONS

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A foliation $\mathscr{F}$ of a manifold V is said to be C^s-structurally stable (or simply C^s-stable) if there exists a neighborhood U of $\mathscr{F}$, in the C^s-topology, such that for each foliation $\mathscr{F}$ in U, there exists a homeomorphism h of V which sends the leaves of $\mathscr{F}'$ to the leaves of $\mathscr{F}$. We also require that h depend continuously on $\mathscr{F}'$. Stability of vector fields, or one-dimensional foliations, is an extensively studied subject with many applications. For an excellent reference to this subject we recommend [8], [4]. It is very natural to consider vector fields with singularities, and more generally one can study stability properties of foliations with singularities or Haefliger structures. This seems to us an important and difficult subject. We shall make some remarks about stability of Haefliger structures, but for the moment we consider nonsingular foliations.

In this paper we shall study C^1 stability of C^∞ foliations of 3-manifolds. We shall see that many 3-manifolds (S^3 for example) admit no structurally stable foliations whatsoever. We classify the stable foliations of $S^2 \times S^1$; they are relatively simple in structure, and we construct foliations of T^3 which are stable. In § 2, we discuss stability of the intrinsic components of a foliation. We prove there is an open dense set of foliations on any 3-manifold, for which the intrinsic components are stable. This encourages the study of a stratification of the space of foliations. Finally we shall make some remarks on Haefliger structures.

1. Instability on some 3-manifolds

In this section we suppose V is a closed oriented 3-manifold, and $\mathscr{F}$ a transversally oriented foliation of V of codimension one and class C^∞.

Theorem 1.1 (*H. Rosenberg and D. Weil*). *Suppose* $(V\mathscr{F})$ *satisfies condition*

i) *there exists a closed transversal curve to* $\mathscr{F}$ *which is null homotopic in* V, *or*

ii) $\pi_2(V) \neq (0)$ *and* $V \neq S^2 \times S^1$.

Then $\mathscr{F}$ *is* C^1*-instable, i.e., one can approximate* $\mathscr{F}$, *in the* C^1*-topology, by foliations nonconjugate to* $\mathscr{F}$.

Remark. Conditions i) and ii) are the hypotheses necessary to apply

Communicated by S. Smale, January 28, 1974.