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A GENERALIZATION OF THE CLOSED SUBGROUP THEOREM TO QUOTIENTS OF ARBITRARY MANIFOLDS

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1. Introduction

The purpose of this paper is to present a new necessary and sufficient condition for an equivalence relation R on a differentiable manifold M to be regular (i.e., such that the quotient M/R is also a differentiable manifold, and that the canonical projection $M \to M/R$ is a submersion). Our condition is motivated by a problem of system theory. Given a nonlinear system $\dot{x} = f(x, u)$ together with an "output map" $y = \varphi(x)$, one can associate with every "input" $u(t)$, $(0 \leq t \leq T)$, and every "initial state" x^0, an "output" $y(t)$, $(0 \leq t \leq T)$, defined as follows: let $x(t)$ be the solution of $\dot{x}(t) = f(x(t), u(t))$ for which $x(0) = x^0$, and let $y(t) = \varphi(x(t))$. If, for an input u, the outputs which correspond to two initial states x^0 and x^1 are not identical, we say that u *distinguishes* between x^0 and x^1. If there is no input which distinguishes between x^0 and x^1, we say that x^0 and x^1 are *indistinguishable*. If there do not exist states x^0 and x^1 which are indistinguishable but different, we say that the system is *observable*. Given a nonobservable system whose state space is a manifold M, we would like to "make it observable". The obvious way to achieve this is by taking the quotient M/R, where R is the equivalence relation of indistinguishability, and by letting this quotient be the state space of our new system. For this to be possible it is necessary that R be regular. The necessary and sufficient condition given in Serre [2, Part II, Chap. 3, § 12, Theorem 2] is not easy to verify. However, in situations where there is more structure, R turns out to be regular for a different reason. As an example, consider the system

$$(1) \qquad \begin{aligned} \dot{X} &= (A + uB)X\ , \qquad X \in G\ , \\ y &= bX\ . \end{aligned}$$

Here the state space G is a Lie group of $n \times n$ matrices, A and B are matrices in the Lie algebra of G, the inputs are real-valued functions, the variable y takes values in $\boldsymbol{R}^n$ (viewed as a space of row vectors), and $b \in \boldsymbol{R}^n$. For each

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