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COMPLEX PARALLELISABLE MANIFOLDS AND THEIR SMALL DEFORMATIONS

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Introduction

By a complex parallelisable manifold we mean a compact complex manifold with the trivial holomorphic tangent bundle. Wang [8] showed that a complex parallelisable manifold is the quotient space of a simply connected, connected complex Lie group by one of its discrete subgroups.

It is known that if the Lie group corresponding to a parallelisable manifold is semi-simple and does not contain SL(2$\boldsymbol{C}$) as a component, then the first Betti number vanishes and its small deformation is rigid, [2], [5], [6].

In this paper we consider the similar problems in the case where the corresponding Lie group is solvable, and obtain quite different results. We note that a simply connected, connected solvable complex Lie group is biholomorphically equivalent to $\boldsymbol{C}^n$ as a complex manifold where $n = \dim_C G$. If a complex parallelisable manifold has a solvable Lie group as the universal covering, it is called a complex solvable manifold.

In § 1 we summarize some known results and give three lemmas. In § 2 by numerical invariants we classify three-dimensional complex solvable manifolds into four classes III-(1), III-(2), III-(3a), III-(3b), and construct some examples in all cases.

In § 3 we construct Kuranishi families of deformations of three-dimensional complex solvable manifolds constructed in § 2. The base spaces of these Kuranishi families which are reduced complex spaces are irreducible in the cases of III-(2) and III-(3a) but reducible in for case of III-(3b), about which we shall give explicit descriptions.

For a compact complex manifold X we denote by $\mathcal{O}$ and Ω^p the sheaves of germs over X of holomorphic functions and p-forms respectively. Recall $h^{p,q} = \dim_C H^q(X, \Omega^p)$ and $P_m(X) = \dim H^0(X, (\Omega^n)^{\otimes m})$ where $n = \dim_C X$. Also we denote by r, κ and b_i respectively the number of linearly independent closed holomorphic 1-forms, Kodaira dimension of X and the i-th Betti number.

S. Iitaka proposed a problem whether all P_m and κ are deformation invariants [1]. However computing the numerical characters of small deformations obtained in the above examples we have

Theorem 2. *$h^{p,q}$ for $(p, q) \neq (0, 0)$, r, P_m and κ are not necessarily invari-*

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