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INFINITESIMAL RIGIDITY OF SUBMANIFOLDS

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Introduction

When a Riemannian manifold M occurs as a submanifold of another Riemannian manifold $\tilde{M}$, rigidity questions naturally arise. Generally speaking, a rigidity theory enumerates the different ways in which M can be isometrically immersed in $\tilde{M}$. Two immersions are equivalent when they differ by a motion (suitably defined) of the ambient space $\tilde{M}$. Rigidity is the term used to denote uniqueness of the immersion up to equivalence.

Even though the word "rigidity" suggests a resistance to bending, the term is generally used to refer to the following concept: M is rigid as a submanifold of $\tilde{M}$ if whenever r_1 and r_2 are isometric immersions of M into $\tilde{M}$, there exists an isometry φ of $\tilde{M}$ such that $r_2 = \varphi \circ r_1$. A second theory, continuous rigidity, deals with one-parameter families of immersions. The third theory and the subject of this paper is called infinitesimal rigidity. As a prototype we have the classical Liebmann problem stated by Stoker [8] as follows:

Liebmann's problem. A closed convex surface in Euclidean three-space is given. It is to be shown that the only small deformations of it which preserve the line element within terms of second order in the deformation parameter are small rigid motions.

Infinitesimal rigidity is a linearized version of continuous rigidity. It turns out that no surface with a planar piece is infinitesimally rigid. In order to get a solution to Liebmann's problem, it is necessary to assume that the given surface has no planar open set. With this assumption, there is a solution, and a proof of Liebmann's theorem may be found in Efimov [1].

In this paper, we formulate the theory of infinitesimal rigidity for submanifolds in general, and then specialize to the case where the ambient space has constant curvature to obtain some results concerning infinitesimal rigidity of spheres. These results are compared and contrasted with those of the standard rigidity theory.

In this paper, all manifolds and maps are assumed sufficiently differentiable for all computations to make sense. All manifolds are assumed connected. For a basic introduction to the theory of hypersurfaces, we refer the reader to [6].

The following notation will be used throughout. A submanifold S of $\tilde{M}$ consists of a manifold M and an immersion r of M into $\tilde{M}$. The Lie algebra

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