J. DIFFERENTIAL GEOMETRY
10 (1975) 39-48

FIBRE BUNDLES AND THE EULER CHARACTERISTIC

DANIEL HENRY GOTTLIEB

1. Introduction

For any fibre bundle $F \xrightarrow{i} E \xrightarrow{p} B$ there are three important maps: the projection p, the fibre inclusion i, and the evaluation $\omega : \Omega B \rightarrow F$. In this paper we demonstrate formulas for each of these maps involving the Euler-Poincaré number of the fibre.

Let M be a compact topological manifold with possibly empty boundary $\dot{M}$, $\chi(M)$ the Euler-Poincaré number of M, G any space of homeomorphisms of M with a continuous action on M, $\omega : G \rightarrow M$ the evaluation map for some base point, $M \xrightarrow{i} E \xrightarrow{p} B$ any (locally trivial) fibre bundle, and $L \subset B$ a (possibly empty) subcomplex of the CW complex B.

Theorem A. *For connected M and any coefficients*

$$\chi(M)\omega^* = 0 : \tilde{H}^*(M) \rightarrow \tilde{H}^*(G) .$$

Theorem B. *There exists a transfer homomorphism* $\tau : H^*(E, p^{-1}(L)) \rightarrow H^*(B, L)$ *such that* $\tau \circ p^* = \chi(M)1$ *for any coefficients.*

Theorem C. *There exists a transfer homomorphism* $\tau : H_*(B, L) \rightarrow H_*(E, p^{-1}(L))$ *such that* $p_* \circ \tau = \chi(M)1$ *for any coefficients.*

Special cases of Theorem A were discovered by the author in [3] and [4]. Note that B and C reduce to the classical transfer theorem for covering spaces when M is a finite set of points. Borel proved a version of Theorem B for M a closed connected differentiable manifold and $M \xrightarrow{i} E \xrightarrow{p} B$ an "oriented" fibre bundle with structural group acting differentially on M and cohomology groups with fields of coefficients whose characteristics does not divide $\chi(M)$, [2]. This result was improved by the author in [1] and [3].

All these theorems are consequences of the next. Let $\dot{E}$ be the subspace of E consisting of those points of E which are in the boundaries of the fibres containing them. Then $(M, \dot{M}) \xrightarrow{i} (E, \dot{E}) \xrightarrow{p} B$ is a fibre pair. If $\dot{M}$ is empty, then $\dot{E}$ is empty.

Theorem D. *Let* M^n *be orientable and connected, and assume* $\pi_1(B)$ *acts*

Received August 21, 1973. This research was partially supported by a National Science Foundation grant.