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BUNDLE HOMOGENEITY AND HOLOMORPHIC CONNECTIONS

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1. Let $\xi : G \to P \xrightarrow{\pi} M$ be a holomorphic principal fiber bundle with group G, total space P, base space M and projection π. Let $a(M)$ be the Lie algebra of all holomorphic vector fields on M, and let $b(\xi)$ be the space of all R_g invariant elements of $a(P)$. (By R_g we mean the map $R_g : P \to P$ given by $R_g(p) = p^g$.) Let $\pi_* : b(\xi) \to a(M)$ be the obvious projection. We say that ξ is bundle homogeneous if π_* is onto. The purpose of this paper is to study the relation between the bundle homogeneity of ξ and the existence of a holomorphic connection on ξ.

In § 2 we fix notation, and in § 3 we gather together the various definitions of a holomorphic connection and show that they are equivalent. This equivalence is well-known but does not seem to be written down anywhere.

In § 4 we prove

Theorem 4.1. *If ξ has a holomorphic connection, then ξ is bundle homogeneous.*

We also show that the converse of Theorem 4.1 is false in general, but we prove

Theorem 4.5. *Let M be complex parallelizable. Then ξ is bundle homogeneous if and only if ξ admits a holomorphic connection.*

If M is compact, Theorem 4.1 is due to A. Morimoto [9]. In the case where M is a complex torus, Theorem 4.5 was proven independently by Y. Matsushima [6] and S. Murakami [10].

Recall that a real product bundle is a holomorphic principal fiber bundle which admits a C^∞ cross-section [7]. In § 5, we obtain a necessary condition for a real product bundle to be bundle homogeneous. This condition is also sufficient if M is compact (Theorem 5.2), and we also obtain some information about the kernel of π_* in this case.

Since Dolbeault cohomology is not a homotopy invariant (Corollary 6.1), we are able in § 6 to apply the results of the previous sections to construct an example of a real product bundle with (noncompact) Kähler base which does not admit a holomorphic connection. Because there are no topological obstructions on a real product bundle, this example shows that the Atiyah obstruction

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