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A GENERALIZATION OF THE TWO-VERTEX THEOREM FOR SPACE CURVES

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1. The classical four-vertex theorem asserts that for a sufficiently smooth simple closed plane curve the curvature has at least four relative extrema. At such a relative extremum of the curvature, or "vertex", the curve has higher contact with its osculating circle. If such a plane curve now is projected stereographically onto the sphere, a vertex goes into a point at which the image curve has higher contact with the osculating plane, i.e., a point at which the torsion vanishes. The four-vertex theorem thus leads naturally to the question: at how many points does the torsion of a closed space curve vanish?

There are closed space curves with nowhere vanishing torsion, for example, a long coil spring which is bent around and connected together. The four-vertex theorem may be interpreted as saying that for a sufficiently smooth simple closed curve on the sphere the torsion vanishes at at least four points. Barner [2] has shown that the conclusion remains true provided only that the curve is simple and lies on a convex surface. A number of similar theorems are known; see Segre [5], [6], and [7].

In this paper we show that a simple closed space curve, which has no tangent lines meeting the curve again (no "cross tangents") and has a point lying on the boundary of the convex hull of the curve though which passes no line meeting the curve in two other points, has at least two points at which the torsion is zero. In fact, if the curve does not lie in a plane, we show that the torsion changes sign.

Two ideas are used in the proof. One is an analysis of certain singularities of the Gauss secant map, and the other is a projection of the curve onto a plane parallel to a plane of support of the convex hull of the space curve. The condition that the curve has a point lying on the boundary of the convex hull of the curve through which passes no line meeting the curve in two other points is required for the projection. The assumption that X has no cross tangents is used in the Gauss secant map. We define three closed one-dimensional submanifolds using singularities of the Stieltjes function

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