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CONFORMALITY OF RIEMANNIAN MANIFOLDS TO SPHERES

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1. Introduction

Let M be an orientable smooth Riemannian manifold of dimension n with Riemannian metric g_{ij}. Let ∇ be the covariant differentiation operator on M, and K_{hijk}, K_{ij}, r be the Riemann curvature tensor, Ricci curvature tensor, and scalar curvature tensor of M respectively. Let X denote the infinitesimal conformal transformation on M so that we have

$$(1.1) \qquad (\mathscr{L}_x g)_{ij} = \nabla_i X_j + \nabla_j X_i = 2\rho g_{ij} ,$$

where ρ is a function, and $\mathscr{L}_x$ denotes the Lie differentiation with respect to X. Assuming that $\mathscr{L}_x r = 0$ Yano, Obata, Hsiung-Mugridge, Hsiung-Stern (see [1], [2], [6], [8]) have studied the condition for a Riemannian n-manifold M to be conformal to an n-sphere. The purpose of this paper is to relax the condition $\mathscr{L}_x r = 0$ further, that is, to assume $\mathscr{L}_{D\rho}\mathscr{L}_x r = 0$, and to obtain conditions for M to be conformal to an n-sphere where $D\rho$ is the vector field associated with the 1-form $d\rho$. Towards this end we prove the following theorems.

Theorem 1.1. *If a compact orientable smooth Riemannian manifold M of dimension $n > 2$ admitting an infinitesimal conformal transformation X: $\mathscr{L}_x g = 2\rho g$, $\rho \neq$ constant with $\mathscr{L}_{D\rho}\mathscr{L}_x r = 0$ satisfies $\int_M \left(A_{ij}\rho^i\rho^j + \frac{\alpha}{n^2}\mathscr{L}_x\mathscr{L}_{D\rho} r\right)dv \geq 0$ where $A_{ij} = K_{ij} - (\alpha r/n)g_{ij}$ and $\alpha = 1$, then M is conformal to an n-sphere.*

Theorem 1.2. *Let M be an orientable smooth Riemannian manifold of dimension $n > 2$ admitting an infinitesimal conformal transformation X satisfying (1.1) such that $\rho \neq$ constant, and $\mathscr{L}_{D\rho}\mathscr{L}_x r = 0$. Then M is conformal to an n-sphere if $\mathscr{L}_x\mathscr{L}_{D\rho} r \geq 0$ and $\mathscr{L}_x |G|^2 = 0$ where $G_{ij} = K_{ij} - (r/n)g_{ij}$.*

Theorem 1.3. *Let M be an orientable smooth Riemannian manifold of dimension $n > 2$ admitting an infinitesimal conformal transformation X satisfying (1.1) such that $\rho \neq$ constant and $\mathscr{L}_{D\rho}\mathscr{L}_x r = 0$. Then M is conformal to an n-sphere if $\mathscr{L}_x\mathscr{L}_{D\rho} r \geq 0$ and $\mathscr{L}_x |W|^2 = 0$ where W is a tensor defined in* § 2.

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