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ON EINSTEIN METRICS

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Let us consider a compact orientable C^∞ manifold M. If a C^∞ Riemannian metric g is given on M, we get a Riemannian manifold (M, g). Let us consider the set of Riemannian metrics g on M such that

$$\int_M dV = 1 ,$$

where dV is the volume element of (M, g). We denote this set by $\mathscr{M}(M)$.

Consider the integral

$$I(g) = \int_M K dV ,$$

where K is the scalar curvature of (M, g). It is well known that a critical point $\bar{g}$ of $I(g)$ in $\mathscr{M}(M)$ is an Einstein metric. A question then arises that whether a given Einstein metric gives a minimum of $I(g)$ or not. It has been established by M. Berger [1] that there exists some Einstein metric (on some C^∞ manifold) for which both $I(g)$ and $-I(g)$ have nonfinite indices.

The purpose of the present paper is to prove the following main theorem.

Theorem. *Let $I(g)$ be the integral as defined above. Then the index of $I(g)$ and also the index of $-I(g)$ are positive at each critical point.*

In the last paragraph some suggestion is given about the index.

1. Infinitesimal deformations of a Riemannian metric from an Einstein metric

In the following we use local coordinates, and a tensor is expressed in its components with respect to the natural frame. Thus $K_{kji}{}^h$ means the curvature tensor

$$K_{kji}{}^h = \partial_k \begin{Bmatrix} h \\ ji \end{Bmatrix} - \partial_j \begin{Bmatrix} h \\ ki \end{Bmatrix} + \begin{Bmatrix} h \\ kl \end{Bmatrix}\begin{Bmatrix} l \\ ji \end{Bmatrix} - \begin{Bmatrix} h \\ jl \end{Bmatrix}\begin{Bmatrix} l \\ ki \end{Bmatrix} ,$$

where $\begin{Bmatrix} h \\ ji \end{Bmatrix}$ is the Christoffel symbol of the metric g. The Ricci tensor and the scalar curvature are respectively given by

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