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# A RELATIVISTIC VERSION OF THE GAUSS-BONNET FORMULA

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## Introduction

The Gauss-Bonnet formula relates the sum of the exterior angles of a geodesic polygon on a surface to the total Gaussian curvature which the polygon encloses. Thus one obtains such statements as: the sum of the interior angles of a geodesic triangle is $\pi$ if and only if the total curvature enclosed by the triangle is zero.

To develop a version of the formula which applies to surfaces with an indefinite metric requires only a careful definition of a quantity to replace "angle" and a check that the arguments of the definite case remain valid. This is done in §§ 1 and 2.

In § 3 an example is given to indicate the kind of physical quantity which the total Gaussian curvature might measure.

## 1. The flat case

In this section the "pseudo-angle" or "proper velocity" between two vectors in a plane with indefinite metric is defined and some elementary properties listed.

Let $M^2$ denote the space of pairs of real numbers with inner product

$$\langle (a_1, a_2), (b_1, b_2) \rangle = -a_1 a_2 + b_1 b_2 \,. \tag{1}$$

Take the positive orientation of $M^2$ to be that given by the vector space basis $\{e_1 = (1, 0), e_2 = (0, 1)\}$.

Let $\alpha \colon I \to M^2$ be a continuously differentiable curve parametrized with respect to proper time, i.e.,

$$\langle \alpha'(s), \alpha'(s) \rangle = -1, 1, 0 \,. \tag{2}$$

The curve $\alpha$ is called timelike, spacelike or null respectively.

Next define a moving frame $\{T(s), N(s)\}$ on $\alpha$ as follows. Let $\{u_1, u_2\}$ be an orthonormal frame at $\alpha(s)$, and set

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