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THE EXISTENCE OF GEODESICS JOINING TWO GIVEN POINTS

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1. Introduction

Let M be a manifold (always assumed to be C^∞, finite dimensional and without boundary), and let $g = (g_{ij})$ be a riemannian metric on M. Since we shall be varying metrics, the following terminology will be convenient.

Definition. An open domain (=connected open subset) D of M will be said to be *g-connected* if every homotopy class of paths in D joining two given points in D contains a (smooth) geodesic segment whose length is minimum *for that class* of paths lying entirely in D. (This geodesic need not furnish a minimum arc length for the corresponding homotopy class of paths in M. See § 5C for an example.)

We shall also have occasion to speak of the *g-completeness* of M, meaning that M is complete in the riemannian sense with respect to g. A standard result in riemannian geometry asserts that the g-completeness of a manifold implies its g-connectivity.

The purpose of this paper is to present some criteria for the g-connectivity of domains. The search for such criteria is largely motivated by the following problem. Let V be a smooth "potential" on M, and consider the conservative dynamical system

$$\frac{D^2x}{dt^2} + \nabla V = 0 \,. \qquad (*)$$

It is well-known that the trajectories to (*) with total energy h are re-parametrized geodesics with respect to the *Jacobi metric* $\tilde{g}_{ij} = (h - V)g_{ij}$. (See e.g. [6] for a rigorous account of this theorem.) Hence the $\tilde{g}$-connectivity of a domain D implies that every pair of points in D can be joined by a trajectory of (*) with total energy h. But the components of $V^{-1}(-\infty, h)$ are not likely to be $\tilde{g}$-complete, and therefore the standard results of riemannian geometry only guarantee the $\tilde{g}$-connectivity of "small" domains (such as normal balls). We would like to be able to construct $\tilde{g}$-connected domains which are reasonably large and physically meaningful.

Finally, we mention that since the geodesics (or trajectories) whose existence

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