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PARALLEL FRAMINGS AND FOLIATIONS ON PSEUDORIEMANNIAN MANIFOLDS

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This paper is in two main parts. The first part is concerned with the global geometrical and topological structure of a pseudoriemannian m-manifold which is foliated by the integral manifolds of a parallel field of tangent k-planes. The second part deals with the more restrictive situation in which k mutually orthogonal parallel vector fields are defined on the manifold.

The main results depend on the local canonical forms for the metric tensor given by Walker [15], [16]. The basic method is to examine the form of the transformations of local coordinates for the canonical charts of Walker, from which it is possible to obtain strong restrictions on the structure of the leaves of the foliations, the vector fields, and the parent manifold. For other applications of this method, see Robertson [10] and Furness and Willmore [1].

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1. Pseudoriemannian metrics

Let E be a real m-plane bundle over a topological space X, with projection $p: E \to X$. Recall that a *Riemannian metric* ρ in E assigns to each $x \in X$ a symmetric positive-definite bilinear form $\rho_x: E_x \times E_x \to R$, where $E_x = p^{-1}(x)$ is the fibre over x, and the forms ρ_x vary continuously with x.

More generally, if we relax the requirement that ρ_x be positive-definite to the condition that ρ_x be nondegenerate, then ρ is called a *pseudoriemannian metric* in E. The signature of ρ_x is then independent of x if X is connected, and is called the signature of ρ in this case. For convenience, we use the term *signature* to mean the ordered pair $(j, m - j)$, where ρ_x has j negative and $m - j$ positive eigenvalues.

Suppose now that F is a subbundle of E of fibre dimension r such that $\rho_x | F_x = 0$, that is, $\rho_x(\lambda, \mu) = 0$ for all $\lambda, \mu \in F_x$. The orthogonal complement G of F in E is then a subbundle of E of fibre dimension $m - r$ such that $F_x \subset G_x$ for all $x \in X$, with

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