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THE SPHERICAL IMAGES OF CONVEX HYPERSURFACES

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1. Introduction

The primary object of study in this paper is the spherical image of a continuous convex hypersurface in euclidean space. The original motivation for this study comes from differential geometry. Therefore, for the benefit of differential geometers, we first present the principal result of interest in the C^∞ category before discussing the technical theorems in convexity theory. To this end, recall that a subset K of the unit n-sphere S^n is (geodescially) *convex* iff for any $p, q \in K$, at least one of the minimal arcs joining p, q lies in K.

Main theorem. *Let M be a complete noncompact orientable C^∞ hypersurface in $\boldsymbol{R}^{n+1}$ $(n > 1)$ with nonnegative but not identically zero sectional curvature. Let $\gamma: M \to S^n$ be the spherical (Gauss) map. Then the following statements are true:*

(α) *$\gamma(M)$ has a convex closure and a convex interior (relative to S^n). More precisely, there exist a unique totally geodesic k-sphere $S^k \subseteq S^n$ $(2 \leq k \leq n)$ and a unique open convex subset K of S^k such that $K \subseteq \gamma(M) \subseteq \operatorname{cl} K$ (=closure of K). In particular, $\gamma(M)$ lies in a closed hemisphere of S^n.*

(β) *The total curvature of M (cf. Chern-Lashof* [3]*) does not exceed one.*

(γ) *M has infinite volume.*

(δ) *If the sectional curvature of M is everywhere positive, then M is homeomorphic with $\boldsymbol{R}^n$, $\gamma: M \to S^n$ is a diffeomorphism onto an open convex subset of S^n, and coordinates in $\boldsymbol{R}^{n+1}$ may be so chosen that M is tangent to the hyperplane $\{x_{n+1} = 0\}$ at the origin and is the graph of a nonnegative strictly convex function (=Hessian is positive definite everywhere) defined in $\{x_{n+1} = 0\}$. Moreover, for any $c > 0$, $M \cap \{x_{n+1} = c\}$ is diffeomorphic to the $(n-1)$-sphere.*

Of the four assertions above, (α) is the crucial one from which (β)–(δ) follow. Now we would like to describe the complete generalizations of (α) and (δ) in the context of convex hypersurfaces. By definition, a *convex hypersurface M* in $\boldsymbol{R}^{n+1}$ is the full boundary of a closed convex C with interior. (We always assume $C \neq \boldsymbol{R}^{n+1}$.) We recall the definition of the (possibly multi-valued)

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