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QUASI-SYMMETRIC IS LOCALLY SYMMETRIC

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1. Let M be the space of left cosets G/K, where G is a connected Lie group with Lie algebra $\mathfrak{g}$ and K is a closed Lie subgroup with Lie subalgebra $\mathfrak{k}$. For simplicity, assume G acts effectively on M on the left. The canonical projection $\pi: G \to M$ takes $e \in G$ to $o = \pi(e) \in M$, and induces the map $\pi_*: T_eG \to T_0M$. For any $X \in \mathfrak{g}$, let X^* be the global vector field on M generated by the 1-parameter group $\{\exp tX\}$. Then $X_0^* = \pi_* X$, and $X \to X^*$ is an injective Lie algebra anti-homomorphism. Assume M is reductive homogeneous, i.e., there is a given Ad (K)-invariant vector space decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. This defines the canonical connection $\tilde{\nabla}$ on M by $(\tilde{\nabla}_{X^*} Y^*)_0 = -[X, Y]_{\mathfrak{p}}$ for $X, Y \in \mathfrak{p}$. Here as usual we identify $\mathfrak{p}$ and T_0M by the restriction of π_*, and the subscript $\mathfrak{p}$ indicates $\mathfrak{p}$ component.

Now suppose M has a naturally reductive pseudo-Riemannian metric $\langle \, , \rangle$, i.e., the metric is G-invariant and the associated Levi-Civita connection ∇ agrees with the canonical torsionless connection for the reductive decomposition $\mathfrak{g} = \mathfrak{k} \otimes \mathfrak{p}$ (see [4, Chapter 10]). Define tensors T and B on $\mathfrak{p}$ by

$$T(X, Y) = [X, Y]_{\mathfrak{p}} \, , \qquad B(X, Y)Z = [[X, Y]_{\mathfrak{k}}, Z] \, .$$

With our conventions of sign, these are just the negatives of the torsion and curvature for the canonical connection at o. Next define endomorphisms T_X and B_X on $\mathfrak{p}$ by

$$T_X(Y) = T(X, Y) \, , \qquad B_X(Y) = B(X, Y)X \, .$$

Then Chavel [2] has given the following definition (in a slightly more special setting).

Definition. The pseudo-Riemannian manifold $(M, \langle \, , \rangle)$ is said to be quasi-symmetric if T_X and B_X commute for all $X \in \mathfrak{p}$.

There are a number of reasons why it seems plausible to study this class of spaces (see [2, p. 20]) but unfortunately, as we will show, all quasi-symmetric spaces are locally symmetric. On the other hand, one can consider the condition as an easy way to distinguish locally symmetric spaces amongst all naturally reductives, so it still may have some interest.

Remark. In [2], Chavel claims that certain spaces M_α^n are quasi-symmetric.

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