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GEODESIC SYMMETRIES IN SPACES WITH SPECIAL CURVATURE TENSORS

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In [3], the authors initiated a study of Riemannian manifolds whose local geodesic symmetries are divergence-preserving (volume-preserving up to sign). We found an infinite sequence of necessary conditions on the curvature tensor, which are sufficient in the analytic case. These results extend to pseudo-Riemannian manifolds with no essential change in proof.

In this paper, we show that the necessary conditions are satisfied in a broad class of spaces defined by imposing a simple algebraic condition on the first covariant derivative of the curvature tensor. This class includes naturally reductive pseudo-Riemannian spaces. We also consider a family of examples, constructed by N. R. Wallach [9] in another context, which shows that there exist reductive Riemannian homogeneous spaces, whose geodesic symmetries fail to be divergence-preserving, and others which satisfy our first necessary condition but not the second.

In spaces which satisfy our algebraic condition, the necessary conditions reduce to verifying an infinite sequence of combinatorial identities involving sums over k indices, $k = 1, 2, \cdots$. The authors succeeded in verifying these for $k \leq 3$, and wish to thank A. Poritz and, especially, D. Slater for assistance in running computer programs associated with the proof of the case $k = 3$. A proof for general k has now been provided by R. T. Bumby [2]. A purely algebraic corollary guarantees the vanishing of the trace of certain recursively defined compositions of two endomorphisms of a finite dimensional vector space. The authors also thank N. R. Wallach for helpful conversations.

1. Preliminaries

Let X denote a nonzero vector at a point 0 of a C^∞ pseudo-Riemanian manifold M, and define an endomorphism Π of the tangent space $T_0(M)$ by

$$(1) \qquad \Pi(Y) = -R(Y, X)X , \qquad Y \in T_0(M) ,$$

where R denotes the curvature of the canonical torsionless metric connection ∇. Let $\nabla_X^0 \Pi = \Pi$, and define $\nabla_X^i \Pi$, $i = 1, 2, \cdots$, by first extending X to the

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