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# A GENERALIZED HODGE THEORY

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## 1. Introduction

For a given nonsingular vector one-form $\underline{h}$ with vanishing Nijenhuis tensor, there is an associated exterior derivative $d_h$ which satisfies a Poincaré lemma and hence provides an $\underline{h}$-dependent version of de Rham's theorem. The exterior derivative $d_h$ also has an adjoint $\delta_h$ with respect to the usual global inner product. This fact permits one to define a strongly elliptic self-adjoint second order differential operator $\Delta_h$ which is a generalization of the Laplace-Beltrami operator. Consequently one can then obtain a generalization of the classical Hodge decomposition theorem.

## 2. Preliminaries

Let $A$ denote the algebra of $C^\infty$ functions on a compact orientable $n$-dimensional Riemannian manifold $M$ without boundary. Let $E$ denote the $A$-module of differential one-forms on $M$. A vector 1-form $\underline{h} \in \mathrm{End}_A(E)$ induces endmorphisms $h^{(q)} \in \mathrm{End}_A(\bigwedge^p E)$ for any nonnegative integer $q$, and the $h^{(q)}$ are defined by setting $h^{(q)} = 0$ if $q > p$, and

$$h^{(q)}(\varphi^1 \wedge \cdots \wedge \varphi^p) = \frac{1}{(p-q)!\,q!} \sum_\pi |\pi| \{\underline{h}\varphi^{\pi(1)} \wedge \cdots \wedge \underline{h}\varphi^{\pi(q)}\} \wedge \varphi^{\pi(q+1)} \wedge \cdots \wedge \varphi^{\pi(p)}$$

if $0 \leq q \leq p$, where $\varphi^i \in E$, $\pi$ runs through all permutations of $(1, \cdots, p)$ and $|\pi|$ denotes the sign of the permutation. The transformation $h^{(0)}$ is taken to be the identity mapping on $\bigwedge^p E$.

In the case where $q = p \leq n$, the operator $h^{(p)}$ is locally represented by an $\binom{n}{p} \times \binom{n}{p}$ matrix $[h^{(p)}]$ relative to some local basis of $p$-forms. If $[\underline{h}]$ denotes an $n \times n$ matrix which locally represents $\underline{h}$, then it can be shown that $\det [h^{(p)}] = (\det [\underline{h}])^{\binom{n-1}{p-1}}$, and hence $\underline{h}$ is invertible on 1-forms if and only if $h^{(p)}$ is invertible in $p$-forms. This fact will be of use later in this section.

An alternating derivation $d_h \colon \bigwedge E \to \bigwedge E$ is obtained as in [3] from $\underline{h}$ and exterior derivation $d$ by setting $d_h = h^{(1)}d - dh^{(1)}$. Thus when $\underline{h}$ is the identity,

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