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# CLIFFORD MULTIPLICATION AND $f$-STRUCTURES

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## 1. Introduction

M. F. Atiyah [1], [2] has neatly applied Clifford multiplication of exterior forms on (smooth, compact) Riemannian manifolds to certain reduction problems of the structure groups of tangent bundles, and considered Clifford multiplication by orientation forms associated with global splittings of tangent bundles into subbundles, that is, plane fields.

We suppose an $m$-dimensional manifold $M$ admits a (1,1)-tensor solution $f$ of $f^3 + f = 0$ with (constant) rank $2l > 0$, that is, an $f$-structure. One may choose a Riemannian structure $\mathscr{G}$ for $M$ so that $f$ is skew. Thus the tangent bundle $T(M)$ of $M$ splits globally as the sum of ker $f$ and the orthogonal complement ker $f^{\perp}$, on which $f$ induces an almost complex structure. Associated with $f$ and $\mathscr{G}$ is a 2-form $\omega$. The purpose of this paper is to study Clifford multiplication by $\omega$ and the orientation form $(\bigwedge \omega)^l$ of the plane field ker $f^{\perp}$.

The existence of an $f$-structure is, of course, equivalent to the reduction of the structure group of $T(M)$ from $\mathscr{O}(m)$ to $\mathscr{O}(m - 2l) \times \mathscr{U}(l)$. The literature devoted to $f$-structures and related topics is extensive, beginning, it seems, with K. Yano [4].

## 2. Algebraic considerations

First we review Clifford multiplication. Clifford multiplication of cross sections of the exterior algebra $\Lambda$ of $M$ depends upon the choice of Riemannian structure $\mathscr{G}$. We consider $\mathscr{G}$ as extended throughout the tensor algebra of $M$. Right and left Clifford multiplications are algebra homomorphisms from cross sections of $\Lambda$ to function-linear cross sections of Hom $(\Lambda, \Lambda)$. Suppose $\alpha$ is a $p$-form and $\beta$ a $q$-form. Define the adjoint of exterior multiplication $\bigwedge$ as follows. If $p < q$, then $\alpha \vee \beta = 0$. If $p \geq q$, then

$$\alpha \vee \beta = \sum_j \mathscr{G}(\alpha, \beta \wedge \mu_j)\mu_j ,$$

where $\{\mu_j\}$ is a local orthonormal basis of the $p - q$ floor of $\Lambda$. This extends to a global definition of $\alpha \vee \beta$. If $v$ is a 1-form and $\alpha$ is a $p$-form, then define the Clifford product $v \cdot \alpha$ as $v \cdot \alpha = v \wedge \alpha - \alpha \vee v$. If $v_1, \cdots, v_q$ are ortho-

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