J. DIFFERENTIAL GEOMETRY
8 (1972) 623-628

SOME EXAMPLES OF MANIFOLDS OF NONNEGATIVE CURVATURE

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The purpose of this note is to describe some examples of manifolds of non-negative curvature and positive Ricci curvature. Apart from homogeneous spaces, no such examples appear in the literature. Our main tool is the formula of O'Neill [8] for riemannian submersions.

Recall that the map $\pi: M^{n+k} \to N^n$ of riemannian manifolds is called a *riemannian* submersion if

1. π is a differentiable submersion, i.e., for all $m \in M$, rank $d\pi_m = n$,
2. $d\pi | H_m$ is an isometry for all $m \in M$.

Here H_m is the orthogonal complement of the kernel V_m of $d\pi$. If $\bar{X}, \bar{Y}$ are horizontal fields, then the vertical component $[\bar{X}, \bar{Y}]_m^V$ of $[\bar{X}, \bar{Y}](m)$ depends only on $\bar{X}(m), \bar{Y}(m)$. Let $x, y \in N_{\pi(m)}$ be orthonormal, $\bar{x}, \bar{y}$ their horizontal lifts at m, and $K, \bar{K}$ denote sectional curvature. Then the formula of O'Neill says

$$(*) \qquad K(x, y) = \bar{K}(\bar{x}, \bar{y}) + \tfrac{3}{4} \| [\bar{x}, \bar{y}]^V \|^2 .$$

Let $G \times M \to M$ be an action of a Lie group on M such that all orbits are closed and of the same type. Then $\pi: M \to G/M$ is a submersion, and any G-invariant riemannian structure on M induces in an obvious way a riemannian structure on $G \backslash M$ such that π becomes a riemannian submersion. If M has nonnegative curvature, then so does $G \backslash M$.[1]

If G acts on N_1, M_1 freely and properly discontinuously on N_1, then it acts freely and properly discontinuously on $N_1 \times M_1$ by the diagonal action. Hence further examples arise by taking products.

Example 1 (*Associated bundles*). Let $M = G_1 \times M_1$, where G_1 is a Lie group with bi-invariant metric, and M_1 has nonnegative curvature. Suppose $G \subset G_1$ is a closed subgroup which acts on M_1 by isometries. Then $(g_1, m) \to (g_1 \cdot g^{-1}, g_m)$ defines a free properly discontinuous action of G on M. As above, $G \backslash M$ inherits a metric of nonnegative curvature. Topologically, $G \backslash M$ is of course the bundle with fibre M_1 associated to the principal fibration $G \to G_1 \to$

Received December 13, 1972. Work done while the author held a Sloan Fellowship at SUNY, Stony Brook, Fall 1970 and at I. M. P. A., Rio de Janeiro, Brazil under O. A. S. contract No. 382201111011, Spring 1971.

[1] Recently, Gromoll and Meyer [4] have constructed a free action of S^3 on SP(2) which preserves the bi-invariant metric. The quotient is an exotic 7-sphere.