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EINSTEIN METRICS ON PRINCIPAL FIBRE BUNDLES

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Introduction

The following construction of Riemannian metrics on principal fibre bundles is well known. (See B. O'Neill [7] and A. Gray [1].) Let M be a manifold with Riemannian metric ds^2, and $\pi: P \to M$ be a principal fibre bundle over M with structure group G. Let γ be any connection form on P, and let $\langle\ ,\ \rangle$ denote a bi-invariant metric on G. Then $g \equiv \pi^* ds^2 + t^2\langle\gamma, \gamma\rangle$, for any $t > 0$, is a Riemannian metric on P with respect to which π becomes a Riemannian submersion. In [4], S. Kobayashi proved that if (M, ds^2) is a Kaehler-Einstein space of positive scalar curvature, then, for proper choices of P, γ and t, g becomes an Einstein metric. In this paper we generalize this result by showing that the above construction can be used to obtain examples of many homogeneous Einstein spaces.

In the first section we compute the Riemann and Ricci curvature tensors for the metric g. Although the Riemann curvature tensor has been computed in detail for such metrics by B. O'Neill [7] and A. Gray [1], we have repeated it here because it is necessary for our purposes to have an explicit expression for the Ricci tensor. Furthermore, we have carried out the computations in terms of forms on the bundle of frames, a technique considerably different from that used in [1] and [7], and one which can be used to great advantage in the applications considered in the second section.

In the second section we consider a class of principal bundles over homogeneous spaces, which are analogous to the example of the spheres $S^{4n+3} \subseteq Q^{n+1}$, Q = quaternions, represented as the quotient spaces $Sp(n+1)/Sp(n)$. At any point $p \in S^{4n+3}$, the tangent space decomposes into the direct sum of a $4n$-dimensional subspace and a 3-dimensional subspace, each invariant under the linear isotropy representation of $Sp(n)$, and furthermore such that the linear isotropy action on the three dimensional subspace is trivial. Hence, by varying the scale of the metric on this three dimensional subspace, the Riemannian metric on S^{4n+3} is changed in such a way that it remains invariant under the action of $Sp(n+1)$. We show that in many such examples the scale on the trivial-action subspace can be chosen in such a way that the resulting metric is Einsteinian. For example, for S^{4n+3} one choice of scale just gives the

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