THE MORSE INDEX THEOREM IN HILBERT SPACE

K. UHLENBECK

When does the critical point of a calculus of variations problem minimize the integral? The classical result is due to Jacobi, who proved that for a regular problem in one independent variable, the integral is minimized at a solution of the Euler-Lagrange equation up to the first conjugate point but not after. Morse extended the theorem to give a formula for the index of a critical curve in terms of the conjugate points along the curve. This result has since been generalized by Edwards [3], Simons [7] and Smale [8] to systems of higher order, minimal surfaces, and partial differential systems respectively. In this article we present an infinite dimensional proof of a general theorem on the index of a bilinear form in Hilbert space which can be applied to all these cases.

The first section contains the abstract formulation and proof of the main theorem (Theorem 1.11). The second section deals with single integral problems and the third with multiple integral problems. In the applications we assume less differentiability than the previous results.

1. The abstract theorem

Let $H_0 \subset H_1 \subset H_1 = H$ be an increasing family of closed Hilbert spaces in $H$ for $0 \leq t \leq 1$, and $A : H \to \mathbb{R}$ be a $C^2$ function on $H$ with 0 as a critical point. Clearly 0 is also a critical point of $A|H_t = A_t$. The Hessian of $A$ at 0 is the bilinear form

$$B = d^2A(0) : H \otimes H \to \mathbb{R}.$$ 

Also the Hessian of $A_t$ at 0 is $B_t = B|H_t \otimes H_t$.

We will be concerned with the properties of $B$ and $B_t$ only, so that we shall assume that $A(v) = \frac{1}{2}B(v, v)$. We recall that the index of 0 as a critical point of $A$ is the dimension of any maximal subspace on which $B(v, v) < 0$ for $v \neq 0$. We define the two functions:

$i(t) =$ index of $A_t =$ dimension of the maximal subspace of $H_t$ on which $B_t$ is negative,

$j(t) =$ dimension of the maximal subspace on which $B_t$ is nonpositive =

Communicated by R. S. Palais, April 24, 1972.