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# A CONGRUENCE THEOREM FOR CLOSED HYPERSURFACES IN RIEMANN SPACES

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## Introduction

We consider two closed oriented surfaces $F$ and $\bar{F}$ in Euclidean 3-space $E^3$ and a differentiable map $\Phi: F \to \bar{F}$ preserving the orientation. The word differentiable always means differentiable of class $C^\infty$. Furthermore, we assume that the set of points on $F$ where the lines $(p, \bar{p})$, $\bar{p} = \Phi(p)$, are tangent to $F$ does not have inner points. Then the following theorems are known:

A) If all the lines $(p, \bar{p})$ are parallel and $H(p) = \bar{H}(\bar{p})$ ($H$ and $\bar{H}$ are the mean curvatures of $F$ and $\bar{F}$ respectively), then the surface $\bar{F}$ is obtained from $F$ by a single translation, i.e., the distances $p\bar{p}$ are the same for all points $p$ on $F$ (H. Hopf and K. Voss [8]).

B) If all the lines $(p, \bar{p})$ go through a fixed point 0 (which does or does not lie on $F$ or $\bar{F}$) and if $rH(p) = \bar{r}\bar{H}(\bar{p})$ ($r$ and $\bar{r}$ are the distances of $p$ and $\bar{p}$ from 0), then $F$ is obtained from $F$ by a homothety, in other words the ratio $\bar{r}/r$ is constant (A. Aeppli [1]).

In order to generalize these two theorems we consider the following case: Let $R^{n+1}$ be an $(n + 1)$-dimensional Riemann space, and $\Phi(p, s)$ be a one-parameter group of transformations of $R^{n+1}$ into itself. Furthermore, let $F^n$ and $\bar{F}^n$ be two $n$-dimensional hypersurfaces of $R^{n+1}$ such that the points of $\bar{F}^n$ are given by the formula:

$$\bar{p} = \Phi(p, f(p)) , \qquad p \in F^n ,$$

where $f(p)$ is a differentiable function of $F^n$. To generalize the condition for the mean curvatures, we have to introduce an additional family of hypersurfaces, one for every point of $F^n$, given by the formula:

$$\tilde{F}^n_p = \Phi(F^n, f(p)) .$$

Then the point $\bar{p} = \Phi(p, f(p))$ lies on the hypersurfaces $\bar{F}^n$ and $\tilde{F}^n_p$ and we define:

$$\bar{H}(\bar{p}) = \text{mean curvature of } \bar{F}^n \text{ at } \bar{p} ,$$
$$\tilde{H}(\bar{p}) = \text{mean curvature of } \tilde{F}^n_p \text{ at } \bar{p} .$$

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