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ON THE ATIYAH-BOTT FORMULA FOR ISOLATED FIXED POINTS

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Introduction

The original Lefschetz fixed point theorem [11] states that if M is a compact manifold, $f\colon M \to M$ has isolated fixed points and $L(f)$ denotes the Lefschetz number of f, then

$$L(f) = \sum_p \deg_p (1 - f) ,$$

where the sum runs over the fixed points p of f. If f is a smooth map and the fixed point p is *simple* in the sense that $\det(1 - df_p) \neq 0$, then its local index has the infinitesimal description $\deg_p (1 - f) = \operatorname{sign}\det(1 - df_p)$. Atiyah and Bott have shown that Lefschetz theory also makes sense in the context of elliptic complexes. They show in [2] that if f induces a chain map of an elliptic complex E over M and the fixed points of f are simple, then

$$L(f, E) = \sum_p \nu(p) ,$$

where $\nu(p)$ are infinitesimal invariants of f at p. It is natural to ask whether their local index can be explained as a special case of a *cohomological* formula which always makes sense for isolated fixed points, as in the classical theorem.

The purpose of this paper is to present a general approach to fixed point theory which applied to isolated fixed points gives both the Atiyah-Bott formula and cohomological formulas. This method is based on a classical formula of de Rham [14, § 33] which expresses intersection numbers in Riemannian manifolds in terms of the Green kernel. It leads to an integral representation for the Lefschetz number from which the Atiyah-Bott theorem can be derived by some delicate but quite elementary analysis. Moreover, assuming that the Poincaré lemma holds, a cohomological expression for the index of an isolated fixed point can also be derived. For simple fixed points this reduces of course to the infinitesimal description.

The use of de Rham's formula was motivated by the intersection—theoretic proof of the classical theorem. An exposition of this proof in a form which suggests the steps to be taken in the elliptic context is included in the first

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