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EXTREMAL SETS OF p-TH SECTIONAL CURVATURE

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The object of this paper is to study the pointwise behavior of the p-th order sectional curvature function σ of a Riemannian manifold M. At $m \in M, \sigma$ is a real-valued function on the compact Grassmann manifold $\mathscr{G}$ of p-planes in the tangent space M_m of M at m. We shall describe the subsets of $\mathscr{G}$ on which σ assumes its maximum and minimum.

We shall work in the setting of an arbitrary inner product space V and arbitrary integer p $(2 \leq p \leq \dim V)$. A p-th curvature operator is a self-adjoint linear transformation $R\colon \Lambda^p \to \Lambda^p$, where $\Lambda^p = \Lambda^p(V)$ has the inner product induced by that of V. (For example, if $V = M_m$ and p is even, the Riemannian p-th curvature operator R_p as defined by Thorpe [4].) The Grassmann manifold $\mathscr{G}$ of oriented p-planes in V is viewed as a subset of the unit sphere in Λ^p.

For R in the vector space $\mathscr{R}$ of p-th curvature operators, we consider its sectional curvature $\sigma_R\colon \mathscr{G} \to \boldsymbol{R}$ given by

$$\sigma_R(\mathscr{P}) = \langle R(\mathscr{P}), \mathscr{P}\rangle \qquad (\mathscr{P} \in \mathscr{G}) .$$

With respect to the inner product

$$\langle T, U\rangle = \operatorname{tr} T \circ U \qquad (T, U \in \mathscr{R}) ,$$

$\mathscr{R}$ decomposes orthogonally into $\mathscr{S} \oplus \mathscr{B}$, where $\mathscr{S}$ is the span of the Grassmann quadratic p-relations which define $\mathscr{G}$:

$$\mathscr{G} = \{\alpha \in \Lambda^p \mid \|\alpha\| = 1 \text{ and } \langle S(\alpha), \alpha\rangle = 0 \text{ for all } S \in \mathscr{S}\} ,$$

and $\mathscr{B}$ is the subspace of operators satisfying generalized Bianchi identities (for $p = 2$, the usual first Bianchi identity for a Riemannian curvature operator). The first section of this paper is devoted to this decomposition of $\mathscr{R}$. We show that $R \in \mathscr{B}$ and $\sigma_R \equiv 0$ imply $R = 0$. It follows that $\mathscr{S}$ is the set of curvature operators whose sectional curvature is identically zero.

In § 2 we find a basis for $\mathscr{S}$. This gives a reduction of the Grassmann quadratic p-relations to a minimal set of conditions. We believe that this result in exterior algebra is new and so we state it here (cf. Lemma 1.1).

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