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REMARKS ON CONFORMAL TRANSFORMATIONS

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1. Introduction

Let (M_1, g_1) and (M_2, g_2) be two connected riemannian manifolds. A diffeomorphism from M_1 to M_2 will be said to be conformal if the pull back of the metric g_2 is proportional to g_1. For the two-dimensional case, these diffeomorphisms are just holomorphic transformations between the underlying complex structures of M_1 and M_2. For dimension greater than two, under the condition that $(M_1, g_1) = (M_2, g_2)$ various authors have been trying to find conditions for a one-parameter group of conformal transformations to be actually a one-parameter group of isometries. It seems that the basic philosophy for such a phenomena is similar to that of Schur's theorem, which states that if the sectional curvature of a connected riemannian manifold of dimension greater than two is constant at every point, then the manifold has constant curvature; throughout this paper by curvature alone we always mean sectional curvature. To verify this principle, we modify the Schur's type argument to generalize and simplify some known theorems. Some new phenomena are also obtained.

In § 1, using a result of H. Omori we prove that if M is complete with the sectional curvature bounded from below and the scalar curvature bounded above by a negative constant, then every conformal transformation on M preserving the scalar curvature is an isometry. This result was obtained by M. Obata [12] in case M is compact.

In § 2, we prove that if M is einsteinian and dim $M \geq 3$, then either M has constant curvature or every conformal transformation is a homothety. This is true even for pseudoriemannian manifolds. Kulkarni [8] proved this fact under some additional assumption, namely, at a generic point the curvature function (of the grassmannian of two planes) has only nondegenerate critical points. For the four-dimensional case, he assumed the manifold to be nowhere constantly curved. If M is complete, the general result was obtained by Yano and Nagano [16], and Nagano [11]. A special case of [11] was reproved in [8].

In § 3, we study the totally geodesic submanifolds of a conformally flat manifold. Using these results we are able to prove that a nontrivial riemannian product cannot be conformally flat unless both factors have constant curvature.

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